



# Extended Gauge Theories in Euclidean Space with Higher Spin Fields

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## Abstract

The extended Yang-Mills gauge theory in Euclidean space is a renormalizable (by power counting) gauge theory describing a local interacting theory of scalar, vector, and tensor gauge fields (with maximum spin 2). In this article we study the quantum aspects and various generalizations of this model in Euclidean space. In particular the quantization of the pure gauge model in a common class of covariant gauges is performed. We generalize the pure gauge sector by including Majorana fermions in the adjoint representation of the gauge group. We show that the maximum half-integer spin contained in this model in dimension 4 is  $3/2$  and that this model is also supersymmetric. Moreover we develop an extension of this theory so as to include internal gauge symmetries and the coupling to bosonic matter fields.

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# 1 Introduction

The interacting theories of higher spin fields [1] have attracted a great attention mostly because of the relevant role played by the spin-2 field graviton which, it is well known, should couple to any kind of particle. There is no doubt about the existence of higher spin composite particles in nature, a classical example is given by the observed hadronic resonances. However, up to now, no elementary particle with spin higher than 1 has been observed (among these the graviton).

The main theoretical problems which affect the construction of a consistent interacting theory of elementary higher spin fields [2] in Minkowski space can be briefly summarized as follows: one must require the cancellation of all negative-norm states, a cancellation which is performed by means of higher-spin gauge invariances. These local invariances, though, impose too many restrictions (on the interacting terms) which cannot be satisfied in many circumstances. These restrictions can be circumvented by relaxing some basic requirements of quantum field theory. Indeed a general class of consistent interacting higher-spin gauge theories in dimension 4,3, and 2 exists and describes infinitely many fields containing all the spins [3]. In the formulation of these theories, though, in order to implement the gauge symmetries, necessary to eliminate all the negative norm states, infinitely many auxiliary fields must be introduced. This mechanism induces higher derivatives in the interaction terms and these in turn give rise to non-locality [1]. These theories are of interest, however, since they also establish a connection with string models, even though in the latter all the elementary higher spin excitations beyond the graviton are massive [4].

Up to now in Minkowski space, the only consistent local interacting field theories of massless spin higher than  $1/2$  are the usual abelian and non-abelian Yang-Mills (YM) gauge theories for the spin-1 [5], the gravity for the spin-2 [6], and the supergravity theories [7] for both the spin-2 and spin-3/2. In addition if one requires these theories to be also renormalizable then the above list would further shorten since it does not contain the gravitational interactions. At present string theories, where gravity is consistently coupled to matter and gauge fields of any spin [4], are largely believed to play a central role in the solution of all these problems.

In this paper we analyze a gauge theory of higher spin interactions in a flat Euclidean space-time. In this space, in fact, we shall see that it is possible to construct a general class of renormalizable higher spin gauge theories. This theory was first introduced a number of years ago in [8] where an extension to the abelian gauge theory with scalar, vector, and tensor gauge fields was proposed in Euclidean space. This model is described (in a 4-dimensional flat space-time) by a non-abelian  $U(4)$  gauge theory of the YM's type where the connection field takes values in the usual

Clifford algebra of spinors. Here the scalar, vector, and tensor fields can be identified as the components of the gauge connection along the Clifford algebra basis. In four dimensions the content of the maximum spin of the gauge multiplet is a spin-2 and the full interacting lagrangian is renormalizable by power counting. This model, in the full basis of Clifford algebra, contains three spin-2 fields: two of them are in a standard representation of the rotation group and are described by a symmetric traceless tensor of rank two, whereas the remaining one is contained in one of the irreducible representations of a tensor of rank-3 with two antisymmetric indices [8]-[10].

An interesting aspect of this model is that the gauge transformations mix, in a consistent way, different irreducible representations of the space-time rotation group. Here the gauge spin-2 fields do not have the usual coupling to the energy momentum tensor while they do couple to the lowest spin particles in a consistent way.<sup>2</sup> Moreover the free gauge transformations of the standard spin-2 fields coincide with the usual spin-2 gauge transformations of the Fierz-Pauli field [11].

A controversial question is the analytical continuation of this theory to the Minkowski space. We next recall some problems related to this issue that are still open. Since the elements of the gauge group are not invariant under the Lorentz transformations, the question whether or not the model in [8] is forbidden by the Coleman-Mandula no-go theorem [12] might arise. As pointed out in Ref.[10], the model in [8] circumvents the hypothesis of [12]. The main reason for this is that the theorem in [12] applies only to the global symmetries of the  $S$  matrix and does not deal with the local symmetries of the Lagrangian (see Ref.[13] for a detailed discussion on this issue). Moreover, since the present theory is of the YM's type, we should expect the confinement phenomenon to arise. If so, then the physical spectrum will be described by gauge invariant operators, such as for example the hadron states or glueballs in QCD, and this symmetry will not be manifest in the  $S$  matrix of the physical states. However we stress that, in this model, one of the main statements of the Coleman-Mandula theorem, which is the analytical behaviour of the  $S$  matrix, can be directly checked in perturbation theory. In particular one can verify (by means of the analogy with the gluon scatterings) that, in the pure Yang-Mills sector, the tree-level gauge-invariant amplitudes satisfy all the analytical requirements [14].

This theory is well formulated in an Euclidean space where the gauge group is compact and, being a YM's type gauge theory, it should satisfy also the Osterwalder-Schrader axioms [15]. Therefore it is possible to quantize it by means of the standard path integral method applied to gauge theories. However, when this theory is formulated

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<sup>2</sup>We will see that the standard spin-2 fields appearing in this model cannot be identified with standard graviton field, at least in the weak coupling limit.

directly into a Minkowski space (where the gauge group is non-compact), problems with unitarity of the  $S$  matrix might arise because of unwanted ghost states [8],[10]. Nevertheless we stress that the appearance of an indefinite metrics in the Hilbert space, due the non-compactness of the symmetry group, is not always an obstacle for building a consistent theory [16]–[19]. A pioneering study in this direction was started by Lee and Wick in Ref.[16]. Moreover in the literature various non-compact sigma-models with indefinite metrics also appear in some extended supergravities when the reduction to four dimension is considered [17]. The general conclusion for these models is that a unitary  $S$  matrix in the physical subspace can be obtained from a pseudo-unitary  $S$ -matrix in the full Hilbert space [17], [18].

In the present model the analysis of unitarity is a more complicated issue than in the non-compact sigma-models, mainly because the symmetry is local and it is not an internal one. A careful analysis of the unitarity of the  $S$  matrix (in Minkowski space) is still missing for this theory and it would be worth investigating how the unphysical ghost sector could decouple from the physical amplitudes. The clarification of this problem could be helpful for understanding the relation between unitarity and renormalizability of the spin-2 field interactions in Minkowski space. However in the present paper we do not tackle the issue of unitarity and restrict ourselves to the Euclidean space where the gauge group is compact and the theory is consistent.

Recently in Ref.[10] an interesting proposal to include fermions in the model in[8] has been given. Whereas in Ref.[10] only the sub-group  $SO(4)$  is considered, in this article we shall see that it is straightforward to extend these couplings to the larger group  $U(4)$  which was first considered in Ref.[8]. In this work we show that the model in [8] is supersymmetric when Majorana fermions are added to the action in the adjoint representation of the gauge group. We recall that, in Euclidean space, the fields which are usually called Majorana fields do not satisfy any reality condition. Nevertheless in the present model, such as in the standard  $N=1$  supersymmetric gauge theories (in Euclidean space), the reality of the Majorana fields is not needed to verify invariance under supersymmetric transformations [20].

The paper is organized as follows. In section [2] we briefly recall the model proposed in Ref.[8] and analyze the free particle spectrum. In section [3] we quantize this model in a covariant gauge and give the expression for the ghost lagrangian. In section [4], by following the approach of Ref.[10], we generalize the model in [8] by including Majorana fermions in the adjoint representation of the gauge group and analyze its supersymmetric properties. In section [5] we extend the model in [8] so as to include the standard internal gauge symmetries. The expression for the unified gauge lagrangian, which includes the internal  $SU(N)$  gauge group, is given together with the corresponding infinitesimal gauge transformations. In section [6] we study

the coupling of the extended gauge fields with bosonic matter fields and make some remarks on possible couplings with ordinary matter and gauge fields. Finally the last section is devoted to our conclusions.

## 2 Pure Gauge Action

In the model proposed in Ref.[8] the usual abelian gauge transformations have been extended to non-abelian ones which mix fields of different integer spin. As a consequence the elements of the gauge group transform non-trivially under coordinate rotations. In addition the lagrangian, which is invariant under these extended gauge transformations, describes a local interacting gauge theory of higher spin fields. This model has the attractive feature that it is renormalizable by power counting. (Due to the gauge invariance, we believe that the model is also fully renormalizable, however we do not tackle this issue in the present article.) Moreover another interesting characteristic is that the maximum value of the spin contents ( $S$ ) of the gauge multiplet is fixed by the space-time dimension  $d$ ; in particular for  $d = 4$  we have  $S = 2$ .

Before presenting our analysis we briefly recall the model proposed in Ref.[8]. One first considers a spinorial-vectorial field  $\hat{A}_\mu^{ij}(x)$  in Euclidean four dimensional space, where  $i$  and  $j$  are indices which belong to the Dirac spinorial space ( $i, j = 1, \dots, 4$ ). In particular this field is defined to transform under the euclidean coordinate rotation<sup>3</sup>

$$x \rightarrow x'_\mu = \Lambda_\mu^\nu x_\nu, \quad \text{with} \quad \Lambda_\mu^\alpha \Lambda_\alpha^\nu = \delta_\mu^\nu \quad (1)$$

as follows

$$\hat{A}(x)_\mu \rightarrow \hat{A}'_\mu(x') = S(\Lambda) \hat{A}_\nu(x) S^{-1}(\Lambda) \Lambda_\mu^\nu. \quad (2)$$

Note that in (2)  $S(\Lambda)$  is the usual spinorial representation of the rotation group  $O(4)$  which is given by

$$S(\Lambda) = \exp\left(-\frac{i}{4} \sigma_{\mu\nu} \omega^{\mu\nu}\right), \quad \text{with} \quad S(\Lambda) \gamma_\mu S(\Lambda)^{-1}(\Lambda) = \Lambda^\nu_\mu \gamma_\nu, \quad (3)$$

where  $\omega_{\mu\nu}$  is a function<sup>4</sup> of  $\Lambda_{\mu\nu}$ ,  $\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$ , and  $\gamma_\mu$  are the usual Dirac matrices satisfying the relation  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ . In Ref.[8] it was pointed out that it is very

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<sup>3</sup>We use the convention to sum up the same indices and the Euclidean metric is given by  $\delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$ .

<sup>4</sup>In the case of infinitesimal transformations we have  $\Lambda_{\mu\nu} = \delta_{\mu\nu} + \omega_{\mu\nu} + \mathcal{O}(\omega^2)$ , with  $\omega_{\mu\nu} = -\omega_{\nu\mu}$ . The exact relationship between  $\Lambda$  and  $\omega$  can be found, for example, in Ref. [23].

useful to decompose the field  $\hat{A}_\mu$  along the Clifford algebra basis; this is done in the following manner

$$\hat{A}_\mu(x) = A_\mu(x)\mathbf{1}_\Gamma + \bar{A}_\mu(x)\gamma_5 + T_{\mu\alpha}(x)\gamma^\alpha + \bar{T}_{\mu\alpha}(x)\gamma^\alpha\gamma_5 + C_{\mu\alpha\beta}(x)\frac{\sigma^{\alpha\beta}}{\sqrt{2}}, \quad (4)$$

where  $\mathbf{1}_\Gamma$  is the unity matrix in the Clifford algebra. Indeed in this decomposition the indices of the fields which label the Clifford algebra basis are vectorial indices under  $O(4)$  rotations (the reader can easily check this property by means of Eqs.(2) and (3)). This implies that the fields  $\bar{A}_\mu$ ,  $T_{\mu\nu}$  (and analogously  $\bar{T}_{\mu\nu}$ ) and  $C_{\mu\alpha\beta}$ , respectively, transform as a vector and tensors of rank 2 and 3.

It is worth observing that some components of  $\hat{A}_\mu$ , namely  $T_{\mu\alpha}$ ,  $\bar{T}_{\mu\alpha}$ , and  $C_{\mu\alpha\beta}$ , can be further decomposed in irreducible representation of  $O(4)$  as follows

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4}\delta_{\mu\nu}\phi + S_{\mu\nu} + V_{\mu\nu}^A + V_{\mu\nu}^S, \\ \bar{T}_{\mu\nu} &= \frac{1}{4}\delta_{\mu\nu}\bar{\phi} + \bar{S}_{\mu\nu} + \bar{V}_{\mu\nu}^A + \bar{V}_{\mu\nu}^S, \\ C_{\mu\alpha\beta} &= \frac{1}{3}(\delta_{\mu\alpha}B_\beta - \delta_{\mu\beta}B_\alpha) - \frac{1}{6}\epsilon_{\mu\alpha\beta\delta}\bar{B}^\delta + D_{\mu\alpha\beta}^S + D_{\mu\alpha\beta}^A, \end{aligned} \quad (5)$$

where  $\epsilon_{\mu\nu\alpha\beta}$  is the complete antisymmetric tensor. We now give some clarifications about the fields appearing in Eq.(5):  $\phi$  and  $\bar{\phi}$  are scalars,  $B_\beta$  and  $\bar{B}_\beta$  are vectors while  $S^{\mu\nu}$  is a symmetric traceless tensor of rank-2. The fields  $V_{\mu\nu}^S$  and  $V_{\mu\nu}^A$  (and analogously  $\bar{V}_{\mu\nu}^{S,A}$ ) are antisymmetric tensors in the (1,0) and (0,1) representations, respectively. (Note that with the notation (x,y) we refer to the usual  $SU(2) \times SU(2)$  complex spinorial representation of the rotation group  $O(4)$ .) The tensor fields  $D_{\mu\alpha\beta}^S$  and  $D_{\mu\alpha\beta}^A$  belong to the (3/2, 1) and (1, 3/2) representations respectively; they are antisymmetric in the  $\alpha, \beta$  indices and are traceless. Moreover  $V_{\alpha\beta}^{(S,A)}$  and  $D_{\mu\alpha\beta}^{(S,A)}$  satisfy the following self-duality conditions [10]

$$\begin{aligned} V_{\mu\nu}^{(S,A)} &= \pm \frac{1}{2}\epsilon_{\mu\nu}^{\alpha\beta}V_{\alpha\beta}^{(S,A)}, \\ D_{\mu\alpha\beta}^{(S,A)} &= \pm \frac{1}{2}\epsilon_{\alpha\beta}^{\gamma\delta}D_{\mu\gamma\delta}^{(S,A)}, \end{aligned} \quad (6)$$

where the signs (+) and (−) refer to (S) and (A) respectively.<sup>5</sup>

We now see that, by using the self-duality conditions (6), the  $D_{\mu\alpha\beta}^{(S,A)}$  and  $V_{\alpha\beta}^{(S,A)}$  tensor

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<sup>5</sup>We recall that in Minkowski space it is not possible to impose self-dual (or antiself-dual) conditions on real fields. The self- (or antiself) duality conditions can be imposed only on complex fields. Therefore in Minkowski space each combination  $V_{\mu\nu}^S + V_{\mu\nu}^A$  (and analogously the  $\bar{V}_{\mu\nu}^{(S,A)}$  fields) and  $D_{\mu\alpha\beta}^S + D_{\mu\alpha\beta}^A$ , which appear in Eq.(5), are replaced by only one irreducible field representation.

fields have 8 and 3 degrees of freedom, respectively. Note that, if the fields are massive[21], a spin-2 field is contained in each of the tensors  $S_{\mu\nu}$ ,  $\bar{S}_{\mu\nu}$  and  $D_{\mu\alpha\beta}^{(S,A)}$ . On the contrary, if the spin-2 fields are massless, according to the Weinberg theorem[21] only the left-handed and right-handed polarizations are consistently described by the  $D_{\mu\alpha\beta}^S$  and  $D_{\mu\alpha\beta}^A$  fields, respectively. Therefore, if parity is conserved, one may conclude that a massless spin-2 can be described by the reducible field  $D_{\mu\alpha\beta} = D_{\mu\alpha\beta}^S + D_{\mu\alpha\beta}^A$ .

Returning to the model in [8], the succeeding step is to promote the field  $\hat{A}_\mu$  to a gauge connection by requiring that  $\hat{A}_\mu$  transforms under a local gauge transformation  $U(x)$  as follows

$$\hat{A}_\mu^G(x) = U(x)\hat{A}_\mu(x)U^{-1} + \frac{1}{ig}U(x)\partial_\mu U^{-1}(x), \quad (7)$$

where  $U(x)$ , which belongs to  $U(4)$ , acts on the spinorial indices of  $\hat{A}_\mu$ . In [8] it is required that the new gauge field  $\hat{A}_\mu^G$  should transform, under coordinate rotations, as  $\hat{A}_\mu$  in Eq.(2). Because of this requirement the transformation (under coordinate rotations) properties of  $U(x)$

$$U'(x') = S(\Lambda)U(x)S^{-1}(\Lambda) \quad (8)$$

also follow.

Now by means of the exponential representation one can express  $U(x)$  as follows

$$U(x) = \exp \{ i\hat{\epsilon}(x) \}, \quad (9)$$

where

$$\hat{\epsilon}(x) = \epsilon(x)\mathbf{I}_\Gamma + \bar{\epsilon}(x)\gamma_5 + \epsilon_\mu(x)\gamma^\mu + \bar{\epsilon}_\mu(x)\gamma^\mu\gamma_5 + \epsilon_{\mu\nu}(x)\frac{\sigma^{\mu\nu}}{\sqrt{2}}. \quad (10)$$

Note that the indices which label the basis in Eq.(10), due to the transformation in (8) and Eq.(3), transform as vectorial indices under coordinate rotations. Finally the compact expression of the lagrangian  $L_E$ , which is invariant under the gauge transformations (7), is given by

$$\begin{aligned} L_E(x) &= \frac{1}{16}Tr[\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}], \\ \hat{F}_{\mu\nu}(x) &= \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + ig[\hat{A}_\mu, \hat{A}_\nu], \end{aligned} \quad (11)$$

where in the above expression the commutator and the trace are taken on the Clifford algebra. By using the component fields given in (4), the lagrangian in (11) takes the following form

$$L_E = L_0 + gL_1 + g^2L_2, \quad (12)$$



where

$$\begin{aligned}
L_0 = & \frac{1}{2} \left\{ (\partial_\alpha T_{\beta\gamma} \partial^\alpha T^{\beta\gamma} - \partial_\alpha T_{\beta\gamma} \partial^\beta T^{\alpha\gamma}) + (\partial_\alpha \bar{T}_{\beta\gamma} \partial^\alpha \bar{T}^{\beta\gamma} - \partial_\alpha \bar{T}_{\beta\gamma} \partial^\beta \bar{T}^{\alpha\gamma}) \right. \\
& + (\partial_\alpha \bar{A}_\beta \partial^\alpha \bar{A}^\beta - \partial_\alpha \bar{A}_\beta \partial^\beta \bar{A}^\alpha) + (\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha) \\
& \left. + (\partial_\alpha C_{\beta\gamma\delta} \partial^\alpha C^{\beta\gamma\delta} - \partial_\alpha C_{\beta\gamma\delta} \partial^\beta C^{\alpha\gamma\delta}) \right\}, \tag{13}
\end{aligned}$$

$$\begin{aligned}
L_1 = & 2 \left\{ \sqrt{2} [T_{\alpha\beta} C^{\gamma\beta\delta} (\partial_\gamma T^\alpha_\delta - \partial^\alpha T_{\gamma\delta}) + \bar{T}_{\alpha\beta} C^{\gamma\beta\delta} (\partial_\gamma \bar{T}^\alpha_\delta - \partial^\alpha \bar{T}_{\gamma\delta}) \right. \\
& + \bar{A}^\mu [T^{\alpha\beta} (\partial_\alpha \bar{T}_{\mu\beta} - \partial_\mu \bar{T}_{\alpha\beta}) + \bar{T}^{\alpha\beta} (\partial_\mu T_{\alpha\beta} - \partial_\alpha T_{\mu\beta})] \\
& + [\sqrt{2} C_{\alpha\beta\gamma} C^{\delta\gamma\mu} - \frac{1}{\sqrt{2}} (T_{\alpha\beta} T^{\delta\mu} + \bar{T}_{\alpha\beta} \bar{T}^{\delta\mu})] [\partial_\delta C^{\alpha\beta}_\mu - \partial^\alpha C_\delta^{\beta}_\mu] \\
& \left. + T_{\alpha\beta} \bar{T}^{\gamma\beta} (\partial_\gamma \bar{A}^\alpha - \partial^\alpha \bar{A}_\gamma) \right\}, \tag{14}
\end{aligned}$$

and

$$\begin{aligned}
L_2 = & 4 [C_{\alpha\beta\gamma} C^{\alpha\beta\delta} C^{\mu\gamma\nu} C_{\mu\delta\nu} - C_\alpha^\beta C^{\alpha\delta\mu} C_{\nu\beta\delta} C^{\nu\gamma}_\mu] \\
& + T_{\alpha\beta} T^{\alpha\beta} T_{\gamma\delta} T^{\gamma\delta} - T_{\alpha\beta} T^{\alpha\gamma} T_{\delta\gamma} T^{\delta\beta} + \bar{T}_{\alpha\beta} \bar{T}^{\alpha\beta} \bar{T}_{\gamma\delta} \bar{T}^{\gamma\delta} - \bar{T}_{\alpha\beta} \bar{T}^{\alpha\gamma} \bar{T}_{\delta\gamma} \bar{T}^{\delta\beta} \\
& + 4 [T^{\alpha\beta} T_{\alpha\gamma} C^{\delta\gamma\mu} C_{\delta\beta\mu} + T^{\alpha\beta} T_{\gamma\delta} C^{\gamma\delta\mu} C_{\alpha\beta\mu} - 2 T_{\alpha\beta} T_{\gamma\delta} C^{\alpha\delta\mu} C^{\gamma\beta}_\mu] \\
& + 4 [\bar{T}^{\alpha\beta} \bar{T}_{\alpha\gamma} C^{\delta\gamma\mu} C_{\delta\beta\mu} + \bar{T}^{\alpha\beta} \bar{T}_{\gamma\delta} C^{\gamma\delta\mu} C_{\alpha\beta\mu} - 2 \bar{T}_{\alpha\beta} \bar{T}_{\gamma\delta} C^{\alpha\delta\mu} C^{\gamma\beta}_\mu] \\
& + 2 [T_{\alpha\beta} T^{\alpha\gamma} \bar{T}^{\delta\beta} \bar{T}_{\delta\gamma} + T_{\alpha\beta} \bar{T}^{\alpha\beta} T_{\gamma\delta} \bar{T}^{\gamma\delta} - 2 T_{\alpha\beta} T_{\gamma\delta} \bar{T}^{\alpha\delta} \bar{T}^{\gamma\beta}] \\
& + 2 [T_{\alpha\beta} T^{\alpha\beta} \bar{A}_\mu \bar{A}^\mu - T_{\beta\alpha} T^{\delta\alpha} \bar{A}^\beta \bar{A}_\delta + \bar{T}_{\alpha\beta} \bar{T}^{\alpha\beta} \bar{A}_\mu \bar{A}^\mu - \bar{T}_{\beta\alpha} \bar{T}^{\delta\alpha} \bar{A}^\beta \bar{A}_\delta] \\
& + 4\sqrt{2} [2 T_{\alpha\beta} \bar{T}^{\alpha\gamma} C^{\mu\beta}_\gamma \bar{A}_\mu - T_{\alpha\beta} \bar{T}_{\mu\gamma} \bar{A}^\mu C^{\alpha\beta\gamma} - T_{\mu\alpha} \bar{A}^\mu \bar{T}_{\beta\gamma} C^{\beta\alpha\gamma}]. \tag{15}
\end{aligned}$$

The lagrangian in (11) is invariant under the following local infinitesimal transformations

$$\begin{aligned}
\delta A_\mu &= -\frac{1}{g} \partial_\mu \epsilon, \\
\delta \bar{A}_\mu &= -\frac{1}{g} \partial_\mu \bar{\epsilon} + 2(\bar{\epsilon}^\nu T_{\mu\nu} - \epsilon^\nu \bar{T}_{\mu\nu}), \\
\delta T_{\mu\nu} &= -\frac{1}{g} \partial_\mu \epsilon_\nu + 2(\bar{\epsilon} T_{\mu\nu} + \sqrt{2} \epsilon^\alpha C_{\mu\nu\alpha} - \bar{\epsilon}_\nu \bar{A}_\mu + \sqrt{2} T_{\mu\alpha} \epsilon^\alpha_\nu), \\
\delta \bar{T}_{\mu\nu} &= -\frac{1}{g} \partial_\mu \bar{\epsilon}_\nu + 2(-\bar{\epsilon} T_{\mu\nu} + \sqrt{2} \bar{\epsilon}^\alpha C_{\mu\nu\alpha} + \epsilon_\nu \bar{A}_\mu + \sqrt{2} \bar{T}_{\mu\alpha} \epsilon^\alpha_\nu), \\
\delta C_{\mu\alpha\nu} &= -\frac{1}{g} \partial_\mu \epsilon_{\alpha\nu} + \sqrt{2} (\epsilon_\alpha T_{\mu\nu} - \epsilon_\nu T_{\mu\alpha}) + \sqrt{2} (\bar{\epsilon}_\alpha \bar{T}_{\mu\nu} - \bar{\epsilon}_\nu \bar{T}_{\mu\alpha}) + \\
& \quad 2\sqrt{2} (\epsilon^\beta_\nu C_{\mu\alpha\beta} - \epsilon^\beta_\alpha C_{\mu\nu\beta}). \tag{16}
\end{aligned}$$

Clearly the  $A_\mu$  is a free field since it corresponds to the  $U(1)$  gauge connection of  $U(4)$  and the interacting theory is described by the  $SU(4)$  gauge group. Note that the smallest sub-algebra of  $SU(4)$  is given by the  $\sigma_{\mu\nu}$  generators which belong to the algebra of  $SO(4)$ . The smallest gauge lagrangian, which is invariant under the  $SO(4)$  gauge transformations, is given by the terms in (13-15) containing only the  $C_{\mu\alpha\beta}$  field. The latter was considered in Ref.[10].

One interesting aspect of this model is that the lagrangian (12) can be written in terms of the  $O(4)$  irreducible representations by inserting the fields decomposition (5) in the expressions (13)-(15) (see [8]). Then the corresponding infinitesimal gauge transformations for the irreducible fields are obtained by means of the standard decomposition method, as previously shown in Eq.(16). For example, the corresponding infinitesimal gauge transformations for  $\delta\phi$ ,  $\delta S_{\mu\nu}$ ,  $\delta V_{\mu\nu}^{(S,A)}$  are given by

$$\begin{aligned}\delta\phi &= \delta T_\mu{}^\mu, \quad \delta S_{\mu\nu} = \frac{1}{2}(\delta T_{\mu\nu} + \delta T_{\nu\mu}) - \frac{1}{4}\delta_{\mu\nu}\delta T_\alpha{}^\alpha, \\ \delta V_{\mu\nu}^{(S,A)} &= \frac{1}{4}\left(\delta T_{\mu\nu} - \delta T_{\nu\mu} \pm \frac{1}{2}\delta T^{\alpha\beta}\epsilon_{\alpha\beta\mu\nu}\right),\end{aligned}\tag{17}$$

with the obvious generalization for the other fields. The expression for the lagrangian containing only the  $C_{\mu\alpha\beta}$  field, in terms of the irreducible representation, can be found in Ref.[10].

We here analyze the physical degrees of freedom and the free particle spectrum of the model introduced in [8]. In order to do so we restrict our analysis by only considering the free lagrangian  $L_0$  which is invariant under the abelian sector<sup>6</sup> of the gauge transformations. We first consider<sup>7</sup> the free lagrangian  $L_0(T)$  in Eq.(13) which contains only the tensor field  $T_{\mu\nu}$ . From Eq.(16) we see that it is possible to make a gauge transformation which ensures

$$\partial^\mu S_{\mu\nu} = 0 \quad \text{and} \quad \phi = 0.\tag{18}$$

This result is justified in the following manner. The free gauge transformations for the field  $\tilde{S}_{\mu\nu} = S_{\mu\nu} + 1/4\delta_{\mu\nu}\phi$  are given by  $\delta\tilde{S}_{\mu\nu} = \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu$ . Then, by means of these gauge transformations, the first constraint in (18) can be imposed. This constraint, when the on-shell massless equations for  $S_{\mu\nu}$  and  $\phi$  are used, is invariant under a new gauge transformation. This new gauge degree of freedom allows us to

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<sup>6</sup>By abelian sector of the gauge transformations we mean the non-homogeneous terms in Eq.(16) containing only the derivatives.

<sup>7</sup>In this analysis we do not consider the free particle spectrum described by the fields  $A_\mu$  and  $\bar{A}_\mu$  since it is clear from (13) that they describe two massless spin-1 fields.

eliminate the component of the on-shell massless scalar field  $\phi$ . (Note that the gauge transformations for the field  $\tilde{S}_{\mu\nu}$  and the gauge fixing in (18) are the same ones encountered in the spin-2 field of the Fierz-Pauli lagrangian.) By adopting the gauge fixing (18) all the gauge degrees of freedom are used and no new constraint on  $T_{\mu\nu}$  can be imposed. In particular no constraint may be introduced for the antisymmetric tensor field  $T_{\mu\nu}^A = V_{\mu\nu}^S + V_{\mu\nu}^A$ .

Now if the fields  $V_{\mu\nu}^{(S,A)}$ , in the (1,0) and (0,1) representation fields, were massive then they would describe two spin-1 fields. However since they are massless fields, according to the Weinberg theorem [21], only one massless spin-1 field can be associated to the  $(1,0) \oplus (0,1)$  representation, where the right- and left-handed polarization are contained in the (1,0) and (0,1) representation, respectively. Here the two residual degrees of freedom are contained in the longitudinal components. These degrees are associated to the transverse polarizations of the vectorial field  $L_\mu \equiv \partial_\nu T^{\mu\nu}$ . Indeed these polarizations cannot be gauged out in this model. The only components of the reducible tensor  $T_{\mu\nu}$  which can be gauged out correspond to the vector  $\partial_\mu T^{\mu\nu}$ .

As a consequence the physical polarizations of the field  $T_{\mu\nu}$ , in the momentum space  $k$ , are the “transverse” ones  $\epsilon_{\mu\alpha}^{T(S)}(k)$  and  $\epsilon_{\mu\alpha}^{T(A)}(k)$  (where (S) and (A) refer to the symmetric and antisymmetric tensor in  $\mu$  and  $\alpha$ , respectively) and “longitudinal” ones  $\epsilon_{\mu\alpha}^L(k)$  which satisfy the following conditions

$$k^\mu \epsilon_{\mu\alpha}^{T(A)}(k) = 0, \quad k^\mu \epsilon_{\mu\alpha}^{T(S)}(k) = 0, \quad k^\mu \epsilon_{\mu\alpha}^L(k) = 0, \quad (19)$$

note that in the above relations  $\epsilon_{\mu\alpha}^L$  has not a definite symmetry in  $\mu, \alpha$  and  $k^\alpha \epsilon_{\mu\alpha}^L(k) \neq 0$ . From Eq.(19) we have  $\epsilon_{\mu\alpha}^{T(S)}(k)$  and  $\epsilon_{\mu\alpha}^{T(A)}(k)$  each contain only two independent polarizations. On the contrary in  $\epsilon_{\mu\alpha}^L$  there are four independent polarizations, each of which correspond to one massless spin-1 and two massless spin-0. As a result the on-shell  $T_{\mu\nu}$  field describes the following spectrum: one massless spin-2, two massless spin-1, and two massless spin-0. Therefore in total we count  $3 \times \mathbf{2} + 2 \times \mathbf{1} = \mathbf{8}$  degrees of freedom for the on-shell  $T_{\mu\nu}$  field; this result is in agreement with the naive counting based on the gauge degrees of freedom.

We next analyze the particle spectrum described by the free lagrangian  $L_0(C)$  in Eq.(13) which contains only the field  $C_{\mu\alpha\beta}$ . We can use the gauge degrees of freedom

$$C_{\mu\alpha\beta} \rightarrow C_{\mu\alpha\beta} + \partial_\mu \epsilon_{\alpha\beta} \quad (20)$$

in order to set the following transversality constraints on the fields  $D_{\mu\alpha\beta}^{(S,A)}$

$$\partial^\mu D_{\mu\alpha\beta}^{(S,A)} = 0. \quad (21)$$

The results in Ref.[21] enable us to see that in the massless case the  $D_{\mu\alpha\beta}^{(S)}$  which satisfies Eq.(21) describes two degrees of freedom. These degrees of freedom correspond to a right-handed spin-2 and a right-handed spin-1 field. Analogously we can see that the  $D_{\mu\alpha\beta}^{(A)}$  describes the corresponding left-handed ones. Therefore, since in this model parity is conserved, the physical polarizations of the reducible field  $D_{\mu\alpha\beta}^{(S)} + D_{\mu\alpha\beta}^{(A)}$  will describe a massless spin-2 and spin-1 field.

As a result of the gauge-fixing in (21) there are no gauge degrees of freedom available which would enable us to eliminate other components in the vector fields  $B_\mu$  and  $\bar{B}_\mu$ . Thus the field  $B_\mu$  contains 4 independent polarizations. The two transverse one (respect to the three-momentum) correspond to a massless spin-one polarizations whereas the longitudinal ones are associated with massless spin-0 fields. The spectrum for  $\bar{B}_\mu$  is obtained analogously. As a result the on-shell  $C_{\mu\alpha\beta}$  field describes the following massless spectrum: one spin-2, three spin-1, and four spin-0 fields; so in total we count respectively  $4 \times \mathbf{2} + 4 \times \mathbf{1} = \mathbf{12}$  degrees of freedom, in agreement with the naive counting based on the gauge degrees of freedom.

It is clear that the on-shell particle contents of this model is gauge invariant and one can reach the same conclusions on the spectrum by using different choices for the gauge fixing. Finally we note that the spin-0 particles (or longitudinal photons) which appear in the spectrum are strictly connected to the fact that some longitudinal components of the tensor or vector fields can not be gauged out.

### 3 Covariant Quantization

When the model in [8] is quantized in the Euclidean space the negative norm states are absent since the space-time metric is the  $\delta_{\mu\nu}$  and the gauge group is compact. Moreover, due to the compactness of the gauge group, the theory can be quantized by means of the standard path integral method. Clearly the fact that the theory is well defined in Euclidean space it is not enough to guarantee its analytical continuation to the Minkowski one. Even if the Osterwalder-Schrader (OS) axioms are satisfied, and in particular the property of reflection positivity [15] is verified, one cannot use here the reconstruction theorems<sup>8</sup> of Ref.[15]. Indeed, in the proof of these theorems, the gauge group is not changed by the analytical continuation to the Minkowski space. On the contrary in the model in [8], in order to maintain the Lorentz covariance, we must rotate the gauge group  $U(4)$  to the non-compact one  $U(2,2)$  when the

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<sup>8</sup>These theorems guarantee that the Wightman functions (which satisfy the Wightman axioms) can be completely reconstructed from the analytical continuation of the corresponding Schwinger functions, provided that these last one satisfy the axioms in [15].

analytical continuation to Minkowski space is performed. However, as we discussed in the introduction, the presence of extra negative-norm states (induced by the non-compact groups) is not always an obstacle for building a consistent theory [16]–[19]. In particular, for this model, it should be interesting to see if a Lorentz invariant Hilbert subspace, where the theory is unitary and the unphysical states decouple from the physical amplitudes, exists. However in present paper we do not tackle the issue of unitarity in Minkowski space.

Now we analyze the covariant quantization of this model in Euclidean space and in the most common class of covariant gauges. The path integral representation of the generating functional of the Green functions  $W[J]$  can be formally written as

$$W[J] = \int D\hat{A}_\mu D\hat{\eta}^\dagger D\hat{\eta} \exp \left\{ - \int d^4x \left( L_E + L_{GF} + \imath L_{GH} - \text{Tr}(\hat{J}_\mu \hat{A}^\mu) \right) \right\}, \quad (22)$$

where  $L_E$  is the full lagrangian given in Eq.(12) and  $L_{GF}$  and  $L_{GH}$  correspond to the gauge-fixing and the ghost lagrangian, respectively. In the last term the trace is taken on the Clifford algebra and  $\hat{J}_\mu$ , which can be decomposed as  $\hat{A}_\mu$  in Eq.(4), is the source for the gauge field  $\hat{A}_\mu$ .

In the present study we consider the general class of covariant gauges whose gauge fixing lagrangian is given by<sup>9</sup>

$$\begin{aligned} L_{GF} = & \frac{1}{2\xi} [(\partial^\alpha \bar{A}_\alpha)^2 + \partial^\alpha T_{\alpha\nu} \partial_\beta T^{\beta\nu} + \partial^\alpha \bar{T}_{\alpha\nu} \partial_\beta \bar{T}^{\beta\nu} \\ & + \partial^\alpha C_{\alpha\nu\mu} \partial_\beta C^{\beta\nu\mu}]. \end{aligned} \quad (23)$$

In this gauge the free propagators  $P_{AB}$  (in momentum space  $k_\mu$ ) for the fields  $\bar{A}_\mu$ ,  $T_{\mu\nu}$ ,  $\bar{T}_{\mu\nu}$  and  $C_{\mu\alpha\beta}$  are given by

$$\begin{aligned} P_{(\bar{A}_\mu \bar{A}_\nu)} &= \frac{1}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right), \\ P_{(T_{\mu\alpha} T_{\nu\beta})} &= \frac{1}{k^2} \delta_{\alpha\beta} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right), \\ P_{(\bar{T}_{\mu\alpha} \bar{T}_{\nu\beta})} &= P_{(T_{\mu\alpha} T_{\nu\beta})}, \\ P_{(C_{\mu\alpha\gamma} C_{\nu\beta\delta})} &= \frac{1}{k^2} \frac{(\delta_{\alpha\gamma} \delta_{\beta\delta} - \delta_{\beta\gamma} \delta_{\alpha\delta})}{2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right). \end{aligned} \quad (24)$$

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<sup>9</sup>We restrict our analysis by considering only the interacting theory given by the  $SU(4)$  gauge group.

The propagators in the basis of the  $O(4)$  irreducible representations can be easily obtained from Eqs.(24) by means of the standard decomposition methods. Note that the propagators in Eqs.(24) are diagonal in the basis of the  $O(4)$  irreducible representations only in the t'Hooft-Feynman gauge  $\xi = 1$ . However in the standard calculations of the scattering amplitudes it is more convenient to work with the propagators in the basis of the reducible fields  $T_{\mu\nu}, \bar{T}_{\mu\nu}$  and  $C_{\mu\alpha\beta}$  instead of the irreducible ones. By applying the standard methods with the gauge fixing in (23) we obtain the following ghost lagrangian

$$L_{GH} = L_0 + gL_I, \quad (25)$$

where

$$\begin{aligned} L_0 &= \partial^\alpha \bar{\eta}^* \partial_\alpha \bar{\eta} + \partial^\alpha \eta_\nu^* \partial_\alpha \eta^\nu + \partial^\alpha \bar{\eta}_\nu^* \partial_\alpha \bar{\eta}^\nu + \partial^\alpha \eta_{\mu\nu}^* \partial_\alpha \eta^{\mu\nu}, \\ L_I &= 2 \left[ \bar{\eta}^* T_{\mu\nu} \partial^\mu \bar{\eta}^\nu - \bar{\eta}^* \bar{T}_{\mu\nu} \partial^\mu \eta^\nu - \sqrt{2} (\eta_\nu^* T_{\mu\alpha} \partial^\mu \eta^{\nu\alpha} - \eta_\nu^* C^{\mu\nu\alpha} \partial_\mu \eta_\alpha) \right. \\ &\quad + \eta_\nu^* \bar{T}_{\mu\nu} \partial^\mu \bar{\eta} - \eta_\nu^* \bar{A}_\mu \partial^\mu \bar{\eta}^\nu - \sqrt{2} (\bar{\eta}_\nu^* \bar{T}_{\mu\alpha} \partial^\mu \eta^{\nu\alpha} - \bar{\eta}_\nu^* C^{\mu\nu\alpha} \partial_\mu \bar{\eta}_\alpha) \\ &\quad - \bar{\eta}_\nu^* T^{\mu\nu} \partial_\mu \bar{\eta} + \bar{\eta}_\nu^* \bar{A}^\mu \partial_\mu \eta^\nu + \sqrt{2} (\eta_{\alpha\nu}^* T^{\mu\nu} \partial_\mu \eta^\alpha + \eta_{\alpha\nu}^* \bar{T}^{\mu\nu} \partial_\mu \bar{\eta}^\alpha \\ &\quad \left. + 2 \eta_{\alpha\nu}^* C^{\mu\alpha\beta} \partial_\mu \eta^{\beta\nu} \right]. \end{aligned} \quad (26)$$

The ghost multiplet, which appears in Eqs.(26), is composed by the following fields: a complex scalar  $\bar{\eta}$ , two complex vectors  $\eta_\mu, \bar{\eta}_\mu$ , and a complex antisymmetric tensor  $\eta_{\alpha\beta}$ , all of which are Grassman variables. Note that the vectorial ghost fields always appear when gauge spin-2 fields are present, a classical example is the quantum gravity.

Now we briefly discuss the gauge-invariant regularization of this model. Due to the presence of the  $\gamma_5$  matrix (which does not exist in odd dimensions) in the gauge Clifford algebra basis, it is clear that the usual dimensional-regularization is not particularly suitable for this model. In order to solve this problem, in Ref.[10] it is suggested to adopt the operator regularization scheme [22]. Here we want to point out that there exists another possible  $SU(4)$  gauge invariant regularization scheme, which is also non-perturbative, for this theory: this is provided by means of the corresponding Wilson action on the lattice [24]. Indeed on the lattice the Clifford algebra basis is taken in 4 dimensions and the above mentioned problem of  $\gamma_5$  does not exist. Moreover, even though the fermions matter fields are coupled to the gauge connection, the lattice regularization does not spoil the  $SU(4)$  gauge symmetry. Indeed, as we will show later on, the generator corresponding to the gauge transformation containing the  $\gamma_5$  matrix is not connected to the “standard” chiral transformations and so the Wilson term, which is necessary to solve the doubling

problem, does respect the gauge symmetry. Clearly, when fermions are added to the theory, the Wilson term breaks the (global) “standard” chiral symmetry.

## 4 Supersymmetric Extension

In this section we consider the couplings of the gauge field  $\hat{A}_\mu$  to fermion matter fields which are in the adjoint representation of the gauge group. In Ref. [10] these couplings have been proposed, and the smallest gauge sub-group  $SO(4)$  studied. We generalize this approach by considering the larger group  $SU(4)$ . In addition we consider the matter fields to be Majorana fermions and show that in this case the theory is also invariant under global supersymmetric transformations.

In order to implement these couplings we define in Euclidean space the following Majorana fermion multiplet  $\hat{\Psi}_k^{ij}$  where the up indices  $i, j$  and the down index  $k$  are the usual Dirac indices. Before giving the coordinate transformation rules of  $\hat{\Psi}_k^{ij}$  some definitions are in order. In addition to the spinorial representation of the rotation group  $S(\Lambda)$  we introduce another independent spinorial representation  $\bar{S}(\Lambda)$  of the  $O(4)$  rotation. The matrix  $\bar{S}(\Lambda)$ , in terms of a new Clifford algebra basis

$$\bar{\Gamma}_i = \mathbf{1}_{\bar{\Gamma}}, \bar{\gamma}_5, \bar{\gamma}_\mu, \bar{\gamma}_5, i\bar{\gamma}_\mu\bar{\gamma}_5, \bar{\sigma}_{\mu\nu}, \quad (27)$$

(where  $\mathbf{1}_{\bar{\Gamma}}$  is the unity matrix) is assumed to commute with  $S(\Lambda)$  and to have the same representation as  $S(\Lambda)$  (see Eq.(3)). Note that each element of the  $\bar{\Gamma}_i$  basis is assumed to commute with any other element of the Clifford  $\Gamma_i$  basis.

Now we can define the following coordinate transformation properties of  $\hat{\Psi}_k^{ij}$  in Euclidean space, these are

$$\begin{aligned} x \rightarrow x'_\mu &= \Lambda_\mu^\nu x_\nu, \\ \hat{\Psi}_k^{ij}(x) \rightarrow \hat{\Psi}'_k{}^{ij}(x') &= \bar{S}_{km}(\Lambda) \left\{ S^{\dagger ia}(\Lambda) \hat{\Psi}_m^{ab}(x) S^{bj}(\Lambda) \right\} \end{aligned} \quad (28)$$

or in a more compact notation

$$\hat{\Psi}(x) \rightarrow \hat{\Psi}'(x') = \bar{S}(\Lambda) \left\{ S^{-1}(\Lambda) \hat{\Psi}(x) S(\Lambda) \right\} \quad (29)$$

and analogously for the adjoint field  $\bar{\hat{\Psi}}$

$$\bar{\hat{\Psi}}(x) \rightarrow \bar{\hat{\Psi}}'(x') = \left\{ S^{-1}(\Lambda) \bar{\hat{\Psi}}(x) S(\Lambda) \right\} \bar{S}(\Lambda)^{-1}, \quad (30)$$

where the multiplication of  $S(\Lambda)$  and  $\bar{S}(\Lambda)$  matrices acts on the up and down Dirac indices, respectively. Note that the  $S(\Lambda)$  matrix in Eq.(29) is the same matrix appearing in the coordinate transformation rule of the gauge potential in Eq.(2). We now decompose the field  $\hat{\Psi}$  on the same basis  $\Gamma_i$  of the  $\hat{A}_\mu$  field as follows

$$\hat{\Psi}_i^{jk} = \lambda_i(\gamma_5)^{jk} + \lambda_i^\mu(\gamma_\mu)^{jk} + \lambda_{5i}^\mu \imath(\gamma_5\gamma_\mu)^{jk} + \lambda_i^{\mu\nu} \frac{(\sigma_{\mu\nu})^{jk}}{\sqrt{2}}. \quad (31)$$

In the sequel the notation for the spinorial down index “i” in the component fields, appearing in (31), will be suppressed. As a consequence of the coordinate transformations (29), the component fields  $\lambda$ ,  $\lambda_\mu$  (or analogously  $\lambda_{5\mu}$ ), and  $\lambda_{\mu\nu}$  transform in the following manner

$$\begin{aligned} \lambda &\rightarrow \lambda' &= \bar{S}(\Lambda)\lambda, \\ \lambda_\mu &\rightarrow \lambda'_\mu &= \Lambda_\mu^\alpha \bar{S}(\Lambda)\lambda_\alpha, \\ \lambda_{\mu\nu} &\rightarrow \lambda'_{\mu\nu} &= \frac{1}{2} \left( \Lambda_\mu^\alpha \Lambda_\nu^\beta - \Lambda_\mu^\beta \Lambda_\nu^\alpha \right) \bar{S}(\Lambda)\lambda_{\alpha\beta}. \end{aligned} \quad (32)$$

There  $\lambda$  (resp.  $\lambda_\mu$  and  $\lambda_{\mu\nu}$ ) transform as a spin-1/2 a (resp. spin-3/2) field. We now require that  $\lambda_5$ ,  $\lambda^\mu$ ,  $\lambda_5^\mu$ , and  $\lambda^{\mu\nu}$  are anticommuting fields satisfying the condition

$$\lambda^c \equiv C\bar{\lambda}^t = \lambda, \quad (33)$$

where  $\lambda^t$  is the transpose and the charge conjugation matrix  $C$  satisfies the following properties  $C^\dagger C = 1$  and  $C^{-1} = -C$ . In Minkowski space the fields satisfying Eq.(33) are Majorana fields. In analogy with this terminology we will call Majorana fields all the fields in Euclidean space which satisfy the condition (33). However it is important to note that, while in Minkowski space the Majorana fields are real fermion fields, in Euclidean space the Eq.(33) is not a reality condition for the  $\lambda$  field. Indeed in Euclidean space the real fermion fields do not exist [20].<sup>10</sup>

It is interesting to observe that the Majorana fields  $\lambda_i^\mu$  and  $\lambda_i^{\mu\nu}$  can be decomposed into irreducible representations of the  $O(4)$  coordinate transformations as follows<sup>11</sup>

$$\lambda_i^\mu = \frac{\imath}{\sqrt{2}} (\bar{\gamma}^\mu \bar{\gamma}_5)_{ij} \psi_j + \psi_i^\mu,$$

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<sup>10</sup>In the following we will use the notation  $\bar{\lambda}$  for any component fermionic field which transforms as  $\bar{\lambda} \rightarrow \bar{\lambda} S^{-1}$  under coordinate rotations  $O(4)$ . We recall that in Euclidean space, for second-quantized Dirac fields,  $\bar{\lambda}$  is an independent variable, while in the case of fields satisfying the condition (33) we have that  $\bar{\lambda} = \lambda^t C$ .

<sup>11</sup>The spinorial indices “i,j” appearing in (34) have been temporary reintroduced to avoid confusions with the notation, and the same indices are intended to be summed up.



$$\begin{aligned}
\lambda_{5i}^\mu &= \frac{i}{\sqrt{2}}(\bar{\gamma}^\mu \bar{\gamma}_5)_{ij} \psi_{5j} + \psi_{5i}^\mu, \\
\lambda_i^{\mu\nu} &= i\bar{\sigma}_{ij}^{\mu\nu} \xi_j + i(\bar{\gamma}_{ij}^\mu \xi_j^\nu - \bar{\gamma}_{ij}^\nu \xi_j^\mu) + \psi_i^{\mu\nu},
\end{aligned} \tag{34}$$

where (in compact notation)

$$\begin{aligned}
\bar{\gamma}_\mu \psi^\mu &= \bar{\gamma}_\mu \psi_5^\mu = \bar{\gamma}_\mu \xi^\mu = 0, \quad \bar{\gamma}_\mu \psi^{\mu\nu} = 0, \\
\psi^{\mu\nu} &= -\psi^{\nu\mu}.
\end{aligned} \tag{35}$$

The fields  $\psi$ ,  $\psi_5$ ,  $\xi$ , ( resp.  $\psi^\mu$ ,  $\psi_5^\mu$ ,  $\xi^\mu$ ,  $\psi^{\mu\nu}$ ) describe spin-1/2 (resp. spin-3/2) Majorana field. By means of Eq.(32) it is straightforward to prove that the decompositions (34) are  $O(4)$  irreducible. Moreover the property that the fields in the r.h.s. of (34) are Majorana fields follows from fact that  $\lambda$ ,  $\lambda_\mu$ ,  $\lambda_{5\mu}$  and  $\lambda_{\mu\nu}$  satisfy the condition (33). This can be easily checked by applying the Eq. (33) to the decomposition (34).

In order to couple the field  $\hat{\Psi}$  to the gauge connection  $\hat{A}_\mu$  we need to require that, under the gauge transformations  $U(x)$ , the field  $\hat{\Psi}$  transforms as follows

$$\hat{\Psi}_k^{ij}(x) \rightarrow \hat{\Psi}_k^{Gij}(x) = U^{ia}(x) \hat{\Psi}_k^{ab}(x) U^{\dagger bj}(x). \tag{36}$$

As a result the covariant derivative  $\hat{D}_\mu$  acting on  $\hat{\Psi}$  is given by

$$\hat{D}_\mu \hat{\Psi} = \partial_\mu \hat{\Psi} - ig [\hat{\Psi}, \hat{A}_\mu], \tag{37}$$

where the commutator is taken on the  $\Gamma_i$  Clifford algebra basis and  $g$  is the same coupling appearing in the lagrangian (12). Finally the gauge invariant lagrangian  $L_F$  for the fermion sector (in the adjoint representation of the gauge group) is

$$L_F = \frac{1}{8} \left\{ Tr \left( \bar{\Psi} \bar{\gamma}_\mu \partial^\mu \hat{\Psi} \right) - ig Tr \left( \bar{\Psi} \bar{\gamma}_\mu [\hat{\Psi}, \hat{A}^\mu] \right) \right\}, \tag{38}$$

where, in component notations, we have

$$\begin{aligned}
L_F &= L_F^0 + g L_F^I, \\
L_F^0 &= \frac{1}{2} \left\{ \bar{\lambda} \gamma^\mu \partial_\mu \lambda + \bar{\lambda}_\alpha \gamma^\mu \partial_\mu \lambda^\alpha + \bar{\lambda}_{5\alpha} \gamma^\mu \partial_\mu \lambda_5^\alpha + \bar{\lambda}_{\alpha\beta} \gamma^\mu \partial_\mu \lambda^{\alpha\beta} \right\}, \\
L_F^I &= 2\bar{A}^\mu \left( \bar{\lambda}_\alpha \gamma_\mu \lambda_5^\alpha \right) - 2 T^{\mu\alpha} \left[ \left( \bar{\lambda} \gamma_\mu \lambda_{5\alpha} \right) + \sqrt{2} \left( \bar{\lambda}_\delta \gamma_\mu \lambda_\alpha^\delta \right) \right] \\
&\quad + 2 \bar{T}^{\mu\alpha} \left[ \left( \bar{\lambda} \gamma_\mu \lambda_\alpha \right) - \sqrt{2} \left( \bar{\lambda}_{5\delta} \gamma_\mu \lambda_\alpha^\delta \right) \right] \\
&\quad - \sqrt{2} C^{\mu\alpha\beta} \left[ \left( \bar{\lambda}_\alpha \gamma_\mu \lambda_\beta \right) + \left( \bar{\lambda}_{5\alpha} \gamma_\mu \lambda_{5\beta} \right) + 2 \left( \bar{\lambda}_{\alpha\delta} \gamma_\mu \lambda_\beta^\delta \right) \right],
\end{aligned} \tag{39}$$

where the following relations for anticommuting Majorana fields  $\lambda_{a,b}$

$$\bar{\lambda}_a \gamma_\mu \lambda_b = -\bar{\lambda}_b \gamma_\mu \lambda_a, \quad \bar{\lambda}_a \gamma_\mu \gamma_5 \lambda_b = \bar{\lambda}_b \gamma_\mu \gamma_5 \lambda_a, \quad \bar{\lambda}_a \lambda_b = \bar{\lambda}_b \lambda_a \quad (40)$$

were used. Note that in Eq.(39) we have eliminated from the notation the “bar” over the  $\bar{\gamma}_\mu$  matrices since in the following we work only with the component fields along the  $\Gamma_i$  basis. The lagrangian (39) is not written in the components of  $O(4)$  irreducible representation, but this can be easily obtained by taking into account the decompositions (5) and (34). We observe that non-trivial couplings between the irreducible representation of the gauge and fermion fields can arise in the interacting lagrangian  $L_F^I$ . After some straightforward algebraic manipulations, in terms of the fermion irreducible representations (34), the free Lagrangian  $L_F^0$  is, up to total derivatives,

$$\begin{aligned} L_F^0 = & \frac{1}{2} \bar{\lambda} \gamma^\mu \partial_\mu \lambda + \frac{1}{2} \bar{\psi}_\alpha \gamma^\mu \partial_\mu \psi^\alpha + \frac{1}{2} \bar{\psi} \gamma^\mu \partial_\mu \psi - i\sqrt{2} \bar{\psi} \gamma_5 \partial^\mu \psi_\mu + \frac{1}{2} \bar{\psi}_{5\alpha} \gamma^\mu \partial_\mu \psi_5^\alpha \\ & - i\sqrt{2} \bar{\psi}_5 \gamma_5 \partial^\mu \psi_{5\mu} + \frac{1}{2} \bar{\psi}_{\mu\nu} \gamma^\alpha \partial_\alpha \psi^{\mu\nu} - \frac{i}{4} \bar{\xi}^\nu (\partial_\nu \xi + \partial^\mu \psi_{\mu\nu}), \end{aligned} \quad (41)$$

where, in deriving the expression (41), the relations (40) were used.

Now we give the infinitesimal gauge transformations for the fermion fields

$$\begin{aligned} \delta_G \lambda &= 2(\lambda^\alpha \bar{\epsilon}_\alpha - \lambda_5^\alpha \epsilon_\alpha), \\ \delta_G \lambda_\alpha &= 2(-\lambda \bar{\epsilon}_\alpha + \sqrt{2} \lambda^\beta \epsilon_{\beta\alpha} + \lambda_{5\alpha} \bar{\epsilon} - \sqrt{2} \lambda_{\beta\alpha} \epsilon^\beta), \\ \delta_G \lambda_{5\alpha} &= 2(\lambda \epsilon_\alpha + \sqrt{2} \lambda_5^\beta \epsilon_{\beta\alpha} - \lambda_\alpha \bar{\epsilon} - \sqrt{2} \lambda_{\beta\alpha} \bar{\epsilon}^\beta), \\ \delta_G \lambda_{\alpha\beta} &= -\sqrt{2}(\lambda_\alpha \epsilon_\beta + \lambda_{5\alpha} \bar{\epsilon}_\beta + 2\lambda_{\mu\alpha} \epsilon^\mu_\beta) - (\alpha \leftrightarrow \beta). \end{aligned} \quad (42)$$

These transformations, together with the corresponding ones for the gauge fields (16), leave invariant the lagrangian  $L_F$ .

One of the most important issues of the extended gauge theory containing Majorana fermions in the adjoint representation of the gauge group is that the total action  $S^{tot}$

$$S^{tot} = \int d^4x \left\{ -L_E - L_F + \frac{1}{2} (\bar{D}^2 + D_\mu D^\mu + \bar{D}_\mu \bar{D}^\mu + D_{\mu\nu} D^{\mu\nu}) \right\} \quad (43)$$

is invariant under global supersymmetric (SUSY) transformations. Note that in (43) the where  $L_E$  and  $L_F$  are the ones given in Eq.(12) and Eq.(38), respectively, and the auxiliary fields  $\bar{D}$ ,  $D^\mu$ ,  $\bar{D}^\mu$ , and  $D^{\mu\nu}$  (with  $D^{\mu\nu}$  being an antisymmetric tensor) should be added to the lagrangian in order to close the off-shell SUSY algebra. This result is obtained by generalizing the SUSY transformations of the pure N=1 Super YM gauge

theory (with internal gauge group) to our case. In components these transformations are given by

$$\begin{aligned}
\delta_S A_\mu &= \bar{\omega} \gamma_\mu \lambda, & \delta_S T_{\mu\alpha} &= \bar{\omega} \gamma_\mu \lambda_\alpha, \\
\delta_S \bar{T}_{\mu\alpha} &= \bar{\omega} \gamma_\mu \lambda_{5\alpha}, & \delta_S C_{\mu\alpha\beta} &= \bar{\omega} \gamma_\mu \lambda_{\alpha\beta}, \\
\delta_S \lambda &= \left( \frac{i}{2} \sigma^{\mu\nu} \bar{F}_{\mu\nu} + \gamma_5 \bar{D} \right) \omega, & \delta_S \lambda_\alpha &= \left( \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu\alpha} + \gamma_5 D_\alpha \right) \omega, \\
\delta_S \lambda_{5\alpha} &= \left( \frac{i}{2} \sigma^{\mu\nu} \bar{F}_{\mu\nu\alpha} + \gamma_5 \bar{D}_\alpha \right) \omega, & \delta_S \lambda_{\alpha\beta} &= \left( \frac{i}{2} \sigma^{\mu\nu} F_{\mu\nu\alpha\beta} + \gamma_5 D_{\alpha\beta} \right) \omega, \\
\delta_S \bar{D} &= \bar{\omega} \gamma_5 \gamma^\mu \mathcal{D}_\mu \lambda, & \delta_S D_\alpha &= \bar{\omega} \gamma_5 \gamma^\mu \mathcal{D}_\mu \lambda_\alpha, \\
\delta_S \bar{D}_\alpha &= \bar{\omega} \gamma_5 \gamma^\mu \mathcal{D}_\mu \lambda_{5\alpha}, & \delta_S D_{\alpha\beta} &= \bar{\omega} \gamma_5 \gamma^\mu \mathcal{D}_\mu \lambda_{\alpha\beta},
\end{aligned} \tag{44}$$

where  $\omega$  is the SUSY infinitesimal anticommuting parameter and the expressions for the field-strength  $F_{\mu\nu}^a$  are

$$\begin{aligned}
\bar{F}_{\mu\nu} &= \partial_\mu \bar{A}_\nu - 2g \left( T_{\mu\alpha} \bar{T}_\nu{}^\alpha \right) - (\mu \leftrightarrow \nu), \\
F_{\mu\nu\alpha} &= \partial_\mu T_{\nu\alpha} - 2g \left( \sqrt{2} T_\mu{}^\beta C_{\nu\beta\alpha} - \bar{A}_\mu \bar{T}_{\nu\alpha} \right) - (\mu \leftrightarrow \nu), \\
\bar{F}_{\mu\nu\alpha} &= \partial_\mu \bar{T}_{\nu\alpha} - 2g \left( \sqrt{2} \bar{T}_\mu{}^\beta C_{\nu\beta\alpha} + \bar{A}_\mu T_{\nu\alpha} \right) - (\mu \leftrightarrow \nu), \\
F_{\mu\nu\alpha\beta} &= \partial_\mu C_{\nu\alpha\beta} + \sqrt{2}g \left( T_{\mu\alpha} T_{\nu\beta} + \bar{T}_{\mu\alpha} \bar{T}_{\nu\beta} - 2C_{\mu\beta\delta} C_{\nu\alpha}{}^\delta \right) - (\mu \leftrightarrow \nu).
\end{aligned} \tag{45}$$

The covariant derivatives are defined as  $\mathcal{D}_\mu \lambda^a = \partial_\mu \lambda^a + g \Delta_\mu \lambda^a$ , where the expressions for  $\Delta_\mu \lambda^a$  are

$$\begin{aligned}
\Delta_\mu \lambda &= 2 \left( \lambda^\alpha \bar{T}_{\mu\alpha} - \lambda_5^\alpha T_{\mu\alpha} \right), \\
\Delta_\mu \lambda_\alpha &= 2 \left( -\lambda \bar{T}_{\mu\alpha} + \sqrt{2} \lambda^\beta C_{\mu\beta\alpha} + \lambda_{5\alpha} \bar{A}_\mu - \sqrt{2} \lambda_{\beta\alpha} T_\mu{}^\beta \right), \\
\Delta_\mu \lambda_{5\alpha} &= 2 \left( \lambda \bar{T}_{\mu\alpha} + \sqrt{2} \lambda_5^\beta C_{\mu\beta\alpha} - \lambda_\alpha \bar{A}_\mu - \sqrt{2} \lambda_{\beta\alpha} \bar{T}_\mu{}^\beta \right), \\
\Delta_\mu \lambda_{\alpha\beta} &= -\sqrt{2} \left( \lambda_\alpha T_{\mu\beta} + \lambda_{5\alpha} \bar{T}_{\mu\beta} + 2\lambda_\alpha{}^\delta C_{\mu\delta\beta} \right) - (\alpha \leftrightarrow \beta).
\end{aligned} \tag{46}$$

The corresponding gauge or SUSY transformations for the  $O(4)$  irreducible representations (see the r.h.s. of Eq.(34)<sup>12</sup>) can now be simply obtained by decomposing the reducible transformations  $\delta\lambda^\mu, \delta\lambda^{\mu\nu}, \dots$  etc., in Eqs.(42) or (44) as follows

$$\delta\psi = \frac{i}{2\sqrt{2}} \gamma_\mu \gamma_5 (\delta\lambda^\mu), \quad \delta\psi_5 = \frac{i}{2\sqrt{2}} \gamma_\mu \gamma_5 (\delta\lambda^{5\mu}),$$

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<sup>12</sup>We recall that in order to make a comparison (by using the notation of (47)) between Eqs. (34) and (47), the “bar” over the  $\bar{\gamma}_\mu$  matrices in (34) should be eliminated.

$$\begin{aligned}
\delta\psi^\mu &= \delta\lambda^\mu - \frac{i}{\sqrt{2}}\gamma_\mu\gamma_5(\delta\psi), & \delta\psi_5^\mu &= \delta\lambda_5^\mu - \frac{i}{\sqrt{2}}\gamma_\mu\gamma_5(\delta\psi_5), \\
\delta\xi &= -\frac{i}{8}\sigma_{\mu\nu}(\delta\lambda^{\mu\nu}), & \delta\xi_\mu &= -\frac{i}{2}\gamma_\mu[\delta\lambda^{\mu\nu} - i\sigma^{\mu\nu}(\delta\xi)], \\
\delta\psi^{\mu\nu} &= \delta\lambda^{\mu\nu} - i\sigma^{\mu\nu}(\delta\xi) - i[\gamma^\mu(\delta\xi^\nu) - \gamma^\nu(\delta\xi^\mu)].
\end{aligned} \tag{47}$$

Before concluding the present section we underline the characterizing properties of the above introduced supersymmetric extension. The extended gauge symmetry mixes fields of different spin but with the same statistics. On the contrary a SUSY transformation mixes fields of different statistic. Moreover, due to the fact that the maximum spin of the gauge-charges is spin-1 and the SUSY charges have spin-1/2, the transformations which leave invariant the total action, can mix fields of spin-S with fields of spin- $(S \pm \Delta S)$ , where  $\Delta S = 1$  and  $\Delta S = 1/2$  correspond to a gauge and SUSY transformation, respectively. Finally the product of a SUSY and gauge transformation can produce a  $|\Delta S| = 3/2$  spin-transition.

## 5 Internal Symmetries

In this section we generalize the model in [8] so as to include the standard internal gauge symmetries. This is effected by restricting our choice to the compact groups  $SU(N)$ .

This generalization is obtained by considering a Lie group whose algebra is given by the tensorial product of the Clifford and  $U(N)$  algebra bases. It is important to note that the elements given by the tensorial product of the unity matrix of  $SU(N)$  with the Clifford algebra basis are necessary in order to close the algebra. As a consequence we have (in 4 dimensions) a Lie group containing  $16 \times N^2 - 1$  generators, which form the algebra (in Euclidean space) of  $SU(4 \times N)$ . The corresponding elements of the extended algebra (which we call  $Y$ ) containing internal symmetries generators can be represented in the compact form

$$Y_i^{\mathbf{a}} = \{\tilde{\Gamma}_i \otimes \mathbf{1}_{\mathbf{T}}, \tilde{\Gamma}_i \otimes \mathbf{T}^{\mathbf{a}}, \mathbf{1}_{\Gamma} \otimes \mathbf{T}^{\mathbf{a}}\}, \tag{48}$$

where the symbol  $\otimes$  indicates the standard tensorial product between the matrices  $\Gamma_i$  (of the Clifford algebra basis) and  $\mathbf{T}^{\mathbf{a}}$  (of  $SU(N)$ ),  $\mathbf{1}_{\Gamma}$ , and  $\mathbf{1}_{\mathbf{T}}$  are the unity matrix of  $\Gamma_i$  and  $SU(N)$  respectively. By  $\tilde{\Gamma}_i$  we denote any element of the reduced Clifford algebra basis which does not contain  $\mathbf{1}_{\Gamma}$ . The matrices  $\mathbf{T}^{\mathbf{a}}$  satisfy the following commutation and anticommutation rules

$$[\mathbf{T}^{\mathbf{a}}, \mathbf{T}^{\mathbf{b}}] = if^{\mathbf{abc}}\mathbf{T}^{\mathbf{c}}, \quad \{\mathbf{T}^{\mathbf{a}}, \mathbf{T}^{\mathbf{b}}\} = 2C_F\delta^{\mathbf{ab}}\mathbf{1}_{\mathbf{T}} + d^{\mathbf{abc}}\mathbf{T}^{\mathbf{c}}, \quad [\mathbf{T}^{\mathbf{a}}, \Gamma_i] = 0, \tag{49}$$

where the following normalizations are used  $Tr(\mathbf{T}^a \mathbf{T}^b) = 1/2\delta^{ab}$  and  $C_F = 1/(2N)$  for the  $\mathbf{T}^a$  matrices in the fundamental representation. The structure function  $d^{abc}$  is a complete symmetric tensor with null traces  $d^{abc}\delta_{ab} = 0$ .

This  $Y$  algebra was proposed a long time ago in Refs.[25],[26] in the context of the relativistic extensions of the  $SU(6)$  model [27] for the strong interactions (the corresponding Lie structure functions can be found in Ref.[25]). We want to stress that, although there is a coincidence of the symmetry group, there is no analogy between the model in [8] and the model proposed in Refs.[25],[26], and the theoretical inconsistencies [28] which are present in the latter do not affect the theory of the model in [8].

Now we give the generalization of the gauge transformations (7)-(10) by including the internal symmetries generated by the group  $SU(4 \times N)$ . We begin with the unitary gauge group element  $\mathbf{U}(x)$  which is given by

$$\begin{aligned} \mathbf{U}(x) = & \exp \left\{ \hat{\epsilon} \otimes \mathbf{I}_T + \epsilon^a(x) (\mathbf{T}^a \otimes \mathbf{I}_\Gamma) + \bar{\epsilon}^a(x) (\mathbf{T}^a \otimes \gamma_5) + \epsilon_\alpha^a(x) (\mathbf{T}^a \otimes \gamma^\alpha) \right. \\ & \left. + \bar{\epsilon}_\alpha^a(x) (\mathbf{T}^a \otimes i\gamma^\alpha \gamma_5) + \epsilon_{\alpha\beta}^a(x) \left( \mathbf{T}^a \otimes \frac{\sigma^{\alpha\beta}}{\sqrt{2}} \right) \right\}, \end{aligned} \quad (50)$$

where  $\hat{\epsilon}$  is given in Eq.(10). We require that  $\mathbf{U}(x)$  transforms as  $U(x)$  in Eq.(8) under coordinate transformations. Then, due to Eq.(3) and to the fact that the  $SU(N)$  generators commute with the  $O(4)$  transformations  $[\mathbf{T}^a, S(\Lambda)] = 0$ , only the “greek” indices in the  $\epsilon_i$  parameters in (50), transform like vectorial indices under  $O(4)$  coordinate transformations.

For the generalization of the gauge potential  $\hat{\mathbf{A}}_\mu$  we may proceed in the same way. We define  $\hat{\mathbf{A}}_\mu$  to have the same coordinate and gauge transformation properties of  $\hat{A}_\mu$  in (2) and (7) respectively. Then we decompose  $\hat{\mathbf{A}}_\mu$  along the  $Y$  algebra basis in (48) as follows <sup>13</sup>

$$\begin{aligned} \hat{\mathbf{A}}_\mu = & \hat{A}_\mu \otimes \mathbf{I}_T + A_\mu^a(x) (\mathbf{T}^a \otimes \mathbf{I}_\Gamma) + \bar{A}_\mu^a(x) (\mathbf{T}^a \gamma_5 \otimes \mathbf{I}_\Gamma) + T_{\mu\alpha}^a(x) (\mathbf{T}^a \otimes \gamma^\alpha) \\ & + \bar{T}_{\mu\alpha}^a(x) (\mathbf{T}^a \otimes i\gamma^\alpha \gamma_5) + C_{\mu\alpha\beta}^a(x) \left( \mathbf{T}^a \otimes \frac{\sigma^{\alpha\beta}}{\sqrt{2}} \right), \end{aligned} \quad (51)$$

where the expression of  $\hat{A}_\mu$  is given in Eq.(4). As in the case of  $\hat{A}_\mu$ , the “greek” indices of the component fields in (51) transform like vectorial indices under the  $O(4)$  coordinate transformations.

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<sup>13</sup>To avoid confusion note that the symbols  $\mathbf{T}^a$  and  $T_{\mu\nu}^a$  indicate the  $SU(N)$  generators and the tensor fields along the  $\gamma_\nu \mathbf{T}^a$  basis respectively.

The pure YM lagrangian in terms of the  $\hat{\mathbf{A}}_\mu$  field which generalizes (12), so as to include the local internal symmetries, has the same formal expression of (12)

$$\begin{aligned}\mathbf{L}^{\mathbf{IS}}(x) &= \frac{1}{8}Tr[\hat{\mathbf{F}}_{\mu\nu}\hat{\mathbf{F}}^{\mu\nu}], \\ \hat{\mathbf{F}}_{\mu\nu}(x) &= \partial_\mu\hat{\mathbf{A}}_\nu - \partial_\nu\hat{\mathbf{A}}_\mu + ig[\hat{\mathbf{A}}_\mu, \hat{\mathbf{A}}_\nu],\end{aligned}\tag{52}$$

where the commutators and trace have to be evaluated on the  $Y$  algebra. The full expression for the lagrangian  $\mathbf{L}^{\mathbf{IS}}$  in terms of the component fields in (51) as well as the infinitesimal gauge transformations which leave invariant  $\mathbf{L}^{\mathbf{IS}}$  are given in the Appendix.

It is worth noting that, since the  $Y$  algebra contains also the anticommutators of the  $SU(N)$  generators, the couplings between the extended  $\hat{A}_\mu$  gauge fields (4) and the corresponding ones  $\hat{A}_\mu^{\mathbf{a}}$  (along the  $\mathbf{T}^{\mathbf{a}}$  basis) appear in the lagrangian (60-61) (see Appendix). As a consequence, in the gauge lagrangian the complete symmetric structure functions  $d^{\mathbf{abc}}$  of the  $SU(N)$  gauge group appears. The presence of the  $d^{\mathbf{abc}}$  structure functions is one of the most interesting aspect of this generalized model, which has no precedent (as far as we know) in any known gauge theory. Moreover it is remarkable to note that, after rescaling in Eqs.(60-61) the kinetic term (by setting it in the canonical form) of the pure extended gauge fields  $\hat{A}_\mu$ , the couplings between the  $\hat{A}_\mu$  and  $\hat{A}_\mu^{\mathbf{a}}$  gauge fields decrease as  $g/\sqrt{N}$  (in the fundamental representation) in the large  $N$  limit.

The coupling of the gauge field  $\hat{\mathbf{A}}_\mu$  with fermion fields in the adjoint representation can be obtained in a way similar to the one used in section [4] by taking into account the different commutation rules of the  $Y$  algebra. Typically the generalization so as to include the internal symmetries in the SUSY transformations (47) is also straightforward.

## 6 Bosonic Matter Fields

In this section we study the couplings of the gauge fields with the matter fields  $\Sigma$  which transform as the fundamental representation of the extended  $U(4)$  gauge group. We will see that for these kind of fields the only consistent gauge-invariant theories, which are compatible with the spin-statistic theorem, are of bosonic type. In particular, in order to build a coordinate- and  $U(4)$  gauge-invariant lagrangian, the field  $\Sigma$  of lowest spin must be a non-hermitean one  $\Sigma_{ij}$  (where  $i$  and  $j$  are spinorial indices) which transforms, under  $O(4)$  coordinate transformations, as the adjoint of

the spinorial representation of the rotation group

$$\begin{aligned} x \rightarrow x'_\mu &= \Lambda_\mu^\nu x_\nu, \\ \Sigma_{ij}(x) \rightarrow \Sigma'_{ij}(x') &= (S(\Lambda)^{-1}(\Lambda)\Sigma(x)S(\Lambda))_{ij}, \end{aligned} \quad (53)$$

where  $S(\Lambda)$  is defined in Eq.(3) and the standard matrix multiplication is assumed. Therefore the  $\Sigma$  field can be decomposed as follows

$$\Sigma = \mathbf{1}_\Gamma \varphi + \gamma_5 \bar{\varphi} + \gamma_\mu B^\mu + \gamma_\mu \gamma_5 \bar{B}^\mu + \sigma_{\mu\nu} C^{\mu\nu} \quad (54)$$

where  $\varphi(\bar{\varphi})$ ,  $B_\mu(\bar{B}_\mu)$  and  $C_{\mu\nu}$  are complex scalar, vectorial and tensorial fields respectively. Now we define its properties under local gauge transformations. In particular we have

$$\Sigma \rightarrow \Sigma^G = U(x)\Sigma, \quad \Sigma^\dagger \rightarrow \Sigma^{\dagger G} = \Sigma^\dagger U^\dagger(x) \quad (55)$$

where  $U(x)$  is given in Eq.(9) and the standard multiplication between matrices is assumed. Note that the group  $U(4)$  is defined to act on  $\Sigma$  and  $\Sigma^\dagger$  on the *right* and *left* respectively. Finally, the most general gauge-invariant and renormalizable (by power counting) lagrangian, which contains only the  $\Sigma$  field and the gauge connection  $\hat{A}_\mu$ , is given by

$$\begin{aligned} L_\Sigma &= Tr [\Sigma^\dagger D_\mu D^\mu \Sigma] + Tr [\gamma_5 \Sigma^\dagger D_\mu D^\mu \Sigma] + Tr [\sigma^{\mu\nu} \Sigma^\dagger [D_\mu, D_\nu] \Sigma] + \\ &\quad m_1 Tr [\gamma^\mu \Sigma^\dagger D_\mu \Sigma] + m_2 Tr [\gamma^\mu \gamma_5 \Sigma^\dagger D_\mu \Sigma] + \\ &\quad m_3^2 Tr [\Sigma^\dagger \Sigma] + \lambda_1 Tr [(\Sigma^\dagger \Sigma)^2] + \lambda_2 Tr [\gamma_5 (\Sigma^\dagger \Sigma)^2] + h.c., \end{aligned} \quad (56)$$

where the trace is defined on the spinorial indices and the covariant derivative  $D_\mu$

$$D_\mu^{ij} \equiv \partial_\mu \delta^{ij} + ig \hat{A}_\mu^{ij}. \quad (57)$$

transforms, according to Eq.(7), as  $U(x)D_\mu U^\dagger(x)$ . Each single trace in Eq.(56) is invariant under  $O(4)$  coordinate rotations and  $U(4)$  gauge transformations. The coefficients  $m_{1,2,3}$  are independent mass parameters, and  $\lambda_{1,2}$  are independent adimensional coupling constants.

It is worth to note that apart from the first and last three terms in Eq.(56), the other gauge-invariant terms have no counterpart (as far as we know) in any known gauge theory. Moreover we find that a minimal non-trivial lagrangian can be obtained by taking in Eq.(56), for example, just only the terms proportional to  $m_1$  and  $m_3$ . In particular it is not difficult to see that the lagrangian

$$L_\Sigma^{min} = i Tr [\gamma^\mu \Sigma^\dagger D_\mu \Sigma] + m Tr [\Sigma^\dagger \Sigma] + h.c. \quad (58)$$

generates non-trivial dynamics for the component fields in  $\Sigma$ . In particular one can see that, on-shell, the  $B_\mu$  and  $\bar{B}_\mu$  fields (defined in Eq.(54)) are the dynamical ones, while  $C_{\mu\nu}$ ,  $\varphi$ , and  $\bar{\varphi}$  play the role of auxiliary fields and can be eliminated by solving the corresponding algebraic equations.

It should by now be clear that it is not possible to couple directly the extended gauge symmetry on the ordinary spin-1/2 fermion fields. Simply because the free lagrangian of a standard spin-1/2 field is not invariant under global extended gauge transformations  $U(4)$ . However it is possible to couple directly the matter fields  $\Sigma$  to the ordinary gauge fields. This is now shown with an example. We assume that the  $\Sigma$  are charged under an abelian  $U(1)$  interaction. Then we have to add in the covariant derivative  $D_\mu$  of Eq.(57) the  $U(1)$  gauge connection  $A_\mu$  as follows

$$D_\mu^{ij} = \delta^{ij} \partial_\mu + ig \hat{A}_\mu^{ij} + ie A_\mu \delta^{ij}, \quad (59)$$

where  $A_\mu$  is the  $U(1)$  gauge field and  $e$  is the corresponding gauge coupling. We also assume that the standard matter fields are charged under the  $U(1)$  gauge symmetry. In order to have a reasonable decoupling between the extended gauge sector and the ordinary matter fields, the  $\Sigma$  should have a mass scale much higher than ordinary matter fields scales. In particular the decoupling between ordinary particles and the massless gauge or gaugino sector of the extended gauge theory is generated when the massive  $\Sigma$  are integrated out. Therefore, effective couplings between the sectors of the ordinary gauge and the extended gauge fields can be generated by means of higher dimensional operators which are suppressed by the corresponding inverse powers of the typical mass scale of the  $\Sigma$  fields. This mechanism to generate couplings between the observable sector and the light one of the extended gauge symmetry has some analogies with the gauge-mediated mechanisms proposed in soft SUSY breaking models [30].

We stress that this is one of the possible mechanisms to couple the ordinary matter and gauge sector to the extended one. In particular the role of the electromagnetic gauge vector mediation described above could be played by any other field which is a singlet under the extended gauge transformations.

## 7 Conclusions

In this article we analyzed the free particle spectrum of the extended YM gauge theories in Euclidean space. We found that the physical degree of freedoms contained in the pure gauge sector of the  $SU(4)$  group correspond to the spin contents of the following massless fields: 3 spin-2, 8 spin-1 and 8 spin-0. Some of the spin-1 and



spin-0 fields correspond to the longitudinal polarizations of the tensor fields which cannot be gauged out. Moreover we analyzed some quantum aspects by providing the functional quantization in Euclidean space for a general class of covariant gauges.

When Majorana fermion fields in the adjoint representation of the gauge group are added to the action, we found that the theory is also supersymmetric. Moreover the maximum semi-integer spin is  $3/2$  in four dimension and these Rarita-Schwinger fields could be regarded as the supersymmetric partners of the spin-2 fields present in the pure gauge theory.

The generalization so as to include the standard internal symmetries introduces a larger unified group. In this model the corresponding algebra of this group is given by the tensorial product of the Clifford and the internal symmetry algebra. We give the expression of the extended YM lagrangian generalized so as to include an  $SU(N)$  internal group. An interesting consequence of this unified algebra is that, in addition to the antisymmetric structure functions  $f^{abc}$  of the internal  $SU(N)$  group, also the complete symmetric ones  $d^{abc}$  appear in the lagrangian (60-61).

Finally we analyzed the matter fields which transform as the fundamental representation of the extended gauge symmetry group. In this case we found that these fields can only be of the bosonic type. Moreover it is important to stress that new kind of gauge-invariant and renormalizable couplings can be generated by means of these matter fields (see Eq.(56)), couplings which have no counterpart in any known gauge theory.

We conclude the present section by commenting on the possible future developments of this study. It would be worth investigating the analytical continuation of this model in Minkowski space by analysing the unitarity of the  $S$  matrix in perturbation theory. The understanding of this issue could be helpful in clarifying the relation between unitarity and renormalizability in the higher spin interactions. Moreover, other interesting aspects of this model, which we believe are worth investigating, are the instanton solutions, the spontaneous gauge symmetry breaking, and the possible embedding of this theory in a string framework.

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## Appendix

In this appendix we give the expression of the lagrangian  $\mathbf{L}^{\mathbf{IS}}$  in (52) in terms of the components of the gauge potential (51) along the  $Y$  algebra basis. By using the commutation rules of the  $Y$  algebra we obtain the following result for  $\mathbf{L}^{\mathbf{IS}}$ <sup>14</sup>

$$\begin{aligned}\mathbf{L}^{\mathbf{IS}} &= \frac{1}{4} \left\{ \mathbf{F}_{\mu\nu}^{\mathbf{a}} \mathbf{F}_{\mu\nu}^{\mathbf{a}} + \bar{\mathbf{F}}_{\mu\nu}^{\mathbf{a}} \bar{\mathbf{F}}^{\mathbf{a}\mu\nu} + \mathbf{F}_{\mu\nu\alpha}^{\mathbf{a}} \mathbf{F}^{\mathbf{a}\mu\nu\alpha} + \bar{\mathbf{F}}_{\mu\nu\alpha}^{\mathbf{a}} \bar{\mathbf{F}}^{\mathbf{a}\mu\nu\alpha} + \mathbf{F}_{\mu\nu\alpha\beta}^{\mathbf{a}} \mathbf{F}^{\mathbf{a}\mu\nu\alpha\beta} \right\} \\ &+ \frac{1}{4C_F} \left\{ \bar{\mathbf{F}}_{\mu\nu} \bar{\mathbf{F}}^{\mu\nu} + \mathbf{F}_{\mu\nu\alpha} \mathbf{F}^{\mu\nu\alpha} + \bar{\mathbf{F}}_{\mu\nu\alpha} \bar{\mathbf{F}}^{\mu\nu\alpha} + \mathbf{F}_{\mu\nu\alpha\beta} \mathbf{F}^{\mu\nu\alpha\beta} \right\},\end{aligned}\quad (60)$$

where  $C_F$  is defined in (49) and the sum over the repeated indices is assumed. The expressions of the fields strength  $\mathbf{F}_i^{\mathbf{a}}$  are given by

$$\begin{aligned}\mathbf{F}_{\mu\nu}^{\mathbf{a}} &= \partial_\mu A_\nu^{\mathbf{a}} - \partial_\nu A_\mu^{\mathbf{a}} - g f^{\mathbf{abc}} \left[ A_\mu^{\mathbf{b}} A_\nu^{\mathbf{c}} + \bar{A}_\mu^{\mathbf{b}} \bar{A}_\nu^{\mathbf{c}} + T_{\mu\alpha}^{\mathbf{b}} T_\nu^{\mathbf{c}\alpha} \right. \\ &\quad \left. + \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{T}_\nu^{\mathbf{c}\alpha} + C_{\mu\alpha\beta}^{\mathbf{b}} C_\nu^{\mathbf{c}\alpha\beta} \right], \\ \bar{\mathbf{F}}_{\mu\nu} &= \bar{F}_{\mu\nu} - g 2C_F \left[ T_{\mu\alpha}^{\mathbf{a}} \bar{T}_\nu^{\mathbf{a}\alpha} - (\mu \leftrightarrow \nu) \right], \\ \bar{\mathbf{F}}_{\mu\nu}^{\mathbf{a}} &= \partial_\mu \bar{A}_\nu^{\mathbf{a}} - \partial_\nu \bar{A}_\mu^{\mathbf{a}} - g \left\{ f^{\mathbf{abc}} \left[ \bar{A}_\mu^{\mathbf{b}} A_\nu^{\mathbf{c}} - \frac{1}{4} \bar{C}_{\mu\alpha\beta}^{\mathbf{b}} C_\nu^{\mathbf{c}\alpha\beta} \right] \right. \\ &\quad \left. + d^{\mathbf{abc}} \left( T_{\mu\alpha}^{\mathbf{b}} \bar{T}_\nu^{\mathbf{c}\alpha} \right) + 2 \left[ T_{\mu\alpha}^{\mathbf{a}} \bar{T}_\nu^{\mathbf{a}} - \bar{T}_{\mu\alpha}^{\mathbf{a}} T_\nu^{\mathbf{a}} \right] - (\mu \leftrightarrow \nu) \right\}, \\ \mathbf{F}_{\mu\nu\alpha} &= F_{\mu\nu\alpha} + g 2C_F \left[ \bar{A}_\mu^{\mathbf{a}} \bar{T}_{\nu\alpha}^{\mathbf{a}} - \sqrt{2} T_\mu^{\mathbf{a}\beta} C_{\nu\beta\alpha}^{\mathbf{a}} - (\mu \leftrightarrow \nu) \right], \\ \mathbf{F}_{\mu\nu\alpha}^{\mathbf{a}} &= \partial_\mu T_{\nu\alpha}^{\mathbf{a}} - \partial_\nu T_{\mu\alpha}^{\mathbf{a}} - g \left\{ f^{\mathbf{abc}} \left[ T_{\mu\alpha}^{\mathbf{b}} A_\nu^{\mathbf{c}} - \frac{1}{\sqrt{2}} \bar{T}_\mu^{\mathbf{b}\beta} \bar{C}_{\nu\alpha\beta}^{\mathbf{c}} \right] \right. \\ &\quad \left. + d^{\mathbf{abc}} \left[ \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{A}_\nu^{\mathbf{c}} - \sqrt{2} T_\mu^{\mathbf{b}\beta} C_{\nu\alpha\beta}^{\mathbf{c}} \right] + 2 \left[ \bar{T}_{\mu\alpha} \bar{A}_\nu^{\mathbf{a}} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{A}_\nu \right] \right. \\ &\quad \left. - \sqrt{2} \left( C_{\nu\alpha\beta} T_\mu^{\mathbf{a}\beta} + C_{\nu\alpha\beta}^{\mathbf{a}} T_\mu^\beta \right) \right] - (\mu \leftrightarrow \nu) \right\}, \\ \bar{\mathbf{F}}_{\mu\nu\alpha} &= \bar{F}_{\mu\nu\alpha} - g 2C_F \left[ \bar{A}_\mu^{\mathbf{a}} T_{\nu\alpha}^{\mathbf{a}} + \sqrt{2} \bar{T}_\mu^{\mathbf{a}\beta} C_{\nu\beta\alpha}^{\mathbf{a}} - (\mu \leftrightarrow \nu) \right], \\ \bar{\mathbf{F}}_{\mu\nu\alpha}^{\mathbf{a}} &= \partial_\mu \bar{T}_{\nu\alpha}^{\mathbf{a}} - \partial_\nu \bar{T}_{\mu\alpha}^{\mathbf{a}} - g \left\{ f^{\mathbf{abc}} \left[ \bar{T}_{\mu\alpha}^{\mathbf{b}} A_\nu^{\mathbf{c}} + \frac{1}{\sqrt{2}} T_\mu^{\mathbf{b}\beta} \bar{C}_{\nu\alpha\beta}^{\mathbf{c}} \right] \right. \\ &\quad \left. + d^{\mathbf{abc}} \left[ T_{\nu\alpha}^{\mathbf{b}} \bar{A}_\mu^{\mathbf{c}} - \sqrt{2} \bar{T}_\mu^{\mathbf{b}\beta} C_{\nu\alpha\beta}^{\mathbf{c}} \right] + 2 \left[ T_{\nu\alpha} \bar{A}_\mu^{\mathbf{a}} - T_{\mu\alpha}^{\mathbf{a}} \bar{A}_\nu \right] \right. \\ &\quad \left. - \sqrt{2} \left( C_{\nu\alpha\beta} \bar{T}_\mu^{\mathbf{a}\beta} - C_{\mu\alpha\beta}^{\mathbf{a}} \bar{T}_\nu^\beta \right) \right] - (\mu \leftrightarrow \nu) \right\}, \\ \mathbf{F}_{\mu\nu\alpha\beta} &= F_{\mu\nu\alpha\beta} + g \sqrt{2} C_F \left( T_{\mu\alpha}^{\mathbf{a}} T_{\nu\beta}^{\mathbf{a}} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{T}_{\nu\beta}^{\mathbf{a}} - 2 C_{\mu\gamma\alpha}^{\mathbf{a}} C_{\nu\beta}^{\mathbf{a}\gamma} - (\mu \leftrightarrow \nu) \right),\end{aligned}$$

<sup>14</sup>This result, together with some other results in the paper, have been obtained by means of the algebraic manipulation program Form [31].

$$\begin{aligned}
\mathbf{F}_{\mu\nu\alpha\beta}^{\mathbf{a}} &= \partial_\mu C_{\nu\alpha\beta}^{\mathbf{a}} - \partial_\nu C_{\mu\alpha\beta}^{\mathbf{a}} + g \left\{ f^{\mathbf{abc}} \left[ \frac{1}{2} \bar{A}_\mu^{\mathbf{b}} \bar{C}_{\nu\alpha\beta}^{\mathbf{c}} + \frac{1}{\sqrt{2}} T_{\mu\gamma}^{\mathbf{b}} \bar{T}_{\nu\delta}^{\mathbf{c}} \epsilon_{\alpha\beta}^{\gamma\delta} + C_{\mu\beta\alpha}^{\mathbf{b}} A_\nu^{\mathbf{c}} \right. \right. \\
&- (\mu \leftrightarrow \nu)] + d^{\mathbf{abc}} \frac{1}{\sqrt{2}} [T_{\mu\alpha}^{\mathbf{b}} T_{\nu\beta}^{\mathbf{c}} + \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{T}_{\nu\beta}^{\mathbf{c}} - 2C_{\mu\gamma\alpha}^{\mathbf{b}} C_{\nu\beta}^{\mathbf{c}\gamma} - (\alpha \leftrightarrow \beta)] \\
&+ \sqrt{2} [(T_{\mu\alpha}^{\mathbf{a}} T_{\nu\beta} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{T}_{\nu\beta} - 2C_{\mu\gamma\alpha}^{\mathbf{a}} C_{\nu\beta}^{\gamma} - (\mu \leftrightarrow \nu)) - (\alpha \leftrightarrow \beta)] \Big\}, \quad (61)
\end{aligned}$$

where the expressions for  $\bar{F}_{\mu\nu}$ ,  $F_{\mu\nu\alpha}$ ,  $\bar{F}_{\mu\nu\alpha}$ , and  $F_{\mu\nu\alpha\beta}$  are given in (45) and  $\bar{C}_{\mu\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta} C_\mu^{\gamma\delta}$ ,  $\bar{C}_{\mu\alpha\beta}^{\mathbf{a}} \equiv \epsilon_{\alpha\beta\gamma\delta} C_\mu^{\mathbf{a}\gamma\delta}$ . The lagrangian  $\mathbf{L}^{\mathbf{IS}}$  is invariant under the following infinitesimal gauge transformations

$$\begin{aligned}
\delta \mathbf{A}_\mu^{\mathbf{a}} &= -\frac{1}{g} \partial_\mu \epsilon^{\mathbf{a}} + f^{\mathbf{abc}} [\bar{A}_\mu^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}} + A_\mu^{\mathbf{b}} \epsilon^{\mathbf{c}} + T_{\mu\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}\alpha} + \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}\alpha} + C_{\mu\alpha\beta}^{\mathbf{b}} \epsilon^{\mathbf{c}\alpha\beta}], \\
\delta \bar{\mathbf{A}}_\mu &= \delta \bar{A}_\mu - 2C_F [-T_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}^{\mathbf{a}\alpha} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \epsilon^{\mathbf{a}\alpha}], \\
\delta \bar{\mathbf{A}}_\mu^{\mathbf{a}} &= -\frac{1}{g} \partial_\mu \epsilon^{\mathbf{a}} - \left\{ f^{\mathbf{abc}} [-\bar{A}_\mu^{\mathbf{b}} \epsilon^{\mathbf{c}} - A_\mu^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}} + \bar{C}_{\mu\alpha\beta}^{\mathbf{b}} \epsilon^{\mathbf{c}\alpha\beta}] \right. \\
&+ d^{\mathbf{abc}} [-T_{\mu\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}\alpha} + \bar{T}_{\mu\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}\alpha}] + 2 [-T_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}^{\alpha} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \epsilon^{\alpha} - T_{\mu\alpha} \bar{\epsilon}^{\mathbf{a}\alpha} + \bar{T}_{\mu\alpha} \epsilon^{\mathbf{a}\alpha}] \Big\}, \\
\delta \mathbf{T}_{\mu\alpha} &= \delta T_{\mu\alpha} - 2C_F [\bar{A}_\mu^{\mathbf{a}} \bar{\epsilon}_\alpha^{\mathbf{a}} - \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}^{\mathbf{a}} + \sqrt{2} (T_{\mu\beta}^{\mathbf{a}} \epsilon_\alpha^{\mathbf{a}\beta} + C_{\mu\gamma\alpha}^{\mathbf{a}} \epsilon^{\mathbf{a}\gamma})], \\
\delta \mathbf{T}_{\mu\alpha}^{\mathbf{a}} &= -\frac{1}{g} \partial_\mu \epsilon_\alpha^{\mathbf{a}} - \left\{ f^{\mathbf{abc}} \left[ -T_{\mu\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}} - A_\mu^{\mathbf{b}} \epsilon_\alpha^{\mathbf{c}} + \frac{1}{\sqrt{2}} (\bar{C}_{\mu\alpha\delta}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}\delta} + T_{\mu\delta}^{\mathbf{b}} \bar{\epsilon}_\alpha^{\mathbf{c}\delta}) \right] \right. \\
&+ d^{\mathbf{abc}} [\bar{A}_\mu^{\mathbf{b}} \bar{\epsilon}_\alpha^{\mathbf{c}} - \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}} + \sqrt{2} (T_{\mu\delta}^{\mathbf{b}} \epsilon_\alpha^{\mathbf{c}\delta} + C_{\mu\delta\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}\delta})] \\
&+ 2 [\bar{A}_\mu^{\mathbf{a}} \bar{\epsilon}_\alpha + \bar{A}_\mu \bar{\epsilon}_\alpha^{\mathbf{a}} - \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon} - \bar{T}_{\mu\alpha} \bar{\epsilon}^{\mathbf{a}} \\
&+ \sqrt{2} (\bar{T}_{\mu\delta}^{\mathbf{a}} \epsilon_\alpha^{\delta} + T_{\mu\delta} \epsilon_\alpha^{\mathbf{a}\delta} + C_{\mu\delta\alpha}^{\mathbf{a}} \epsilon^{\delta} + C_{\mu\delta\alpha} \epsilon^{\mathbf{a}\delta})] \Big\}, \\
\delta \bar{\mathbf{T}}_{\mu\alpha} &= \delta \bar{T}_{\mu\alpha} - 2C_F [-\bar{A}_\mu^{\mathbf{a}} \bar{\epsilon}_\alpha^{\mathbf{a}} + T_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}^{\mathbf{a}} + \sqrt{2} (\bar{T}_{\mu\beta}^{\mathbf{a}} \epsilon_\alpha^{\mathbf{a}\beta} + C_{\mu\gamma\alpha}^{\mathbf{a}} \bar{\epsilon}^{\mathbf{a}\gamma})], \\
\delta \bar{\mathbf{T}}_{\mu\alpha}^{\mathbf{a}} &= -\frac{1}{g} \partial_\mu \bar{\epsilon}_\alpha^{\mathbf{a}} - \left\{ f^{\mathbf{abc}} \left[ -\bar{T}_{\mu\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}} - A_\mu^{\mathbf{b}} \bar{\epsilon}_\alpha^{\mathbf{c}} - \frac{1}{\sqrt{2}} (\bar{C}_{\mu\alpha\delta}^{\mathbf{b}} \epsilon^{\mathbf{c}\delta} + T_{\mu\delta}^{\mathbf{b}} \bar{\epsilon}_\alpha^{\mathbf{c}\delta}) \right] \right. \\
&+ d^{\mathbf{abc}} [-\bar{A}_\mu^{\mathbf{b}} \epsilon_\alpha^{\mathbf{c}} + T_{\mu\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}} + \sqrt{2} (\bar{T}_{\mu\delta}^{\mathbf{b}} \epsilon_\alpha^{\mathbf{c}\delta} + C_{\mu\delta\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}\delta})] \\
&+ 2 [-\bar{A}_\mu^{\mathbf{a}} \epsilon_\alpha - \bar{A}_\mu \epsilon_\alpha^{\mathbf{a}} + T_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon} + T_{\mu\alpha} \bar{\epsilon}^{\mathbf{a}} \\
&+ \sqrt{2} (\bar{T}_{\mu\delta}^{\mathbf{a}} \epsilon_\alpha^{\delta} + \bar{T}_{\mu\delta} \epsilon_\alpha^{\mathbf{a}\delta} + C_{\mu\delta\alpha}^{\mathbf{a}} \bar{\epsilon}^{\delta} + C_{\mu\delta\alpha} \bar{\epsilon}^{\mathbf{a}\delta})] \Big\}, \\
\delta \mathbf{C}_{\mu\alpha\beta} &= \delta C_{\mu\alpha\beta} - 2C_F \left\{ \frac{1}{\sqrt{2}} (T_{\mu\alpha}^{\mathbf{a}} \epsilon_\beta^{\mathbf{a}} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}_\beta^{\mathbf{a}}) - \sqrt{2} C_{\mu\delta\alpha}^{\mathbf{a}} \epsilon_\beta^{\mathbf{a}\delta} - (\alpha \leftrightarrow \beta) \right\}, \\
\delta \mathbf{C}_{\mu\alpha\beta}^{\mathbf{a}} &= -\frac{1}{g} \partial_\mu \epsilon_{\alpha\beta}^{\mathbf{a}} - \left\{ f^{\mathbf{abc}} \left[ C_{\mu\beta\alpha}^{\mathbf{b}} \epsilon^{\mathbf{c}} + \frac{1}{2} (-\bar{C}_{\mu\beta\alpha}^{\mathbf{b}} \bar{\epsilon}^{\mathbf{c}} + \bar{A}_\mu^{\mathbf{b}} \bar{\epsilon}_{\alpha\beta}^{\mathbf{c}}) - A_\mu^{\mathbf{b}} \epsilon_{\alpha\beta}^{\mathbf{c}} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\sqrt{2}} \left( T_{\mu\delta}^{\mathbf{b}} \bar{\epsilon}_{\gamma}^{\mathbf{c}} - \bar{T}_{\mu\delta}^{\mathbf{b}} \epsilon_{\gamma}^{\mathbf{c}} \right) \epsilon_{\alpha\beta}^{\delta\gamma} \Big] \\
& + d^{\mathbf{abc}} \left[ \frac{1}{\sqrt{2}} \left( T_{\mu\alpha}^{\mathbf{b}} \epsilon_{\beta}^{\mathbf{c}} + \bar{T}_{\mu\alpha}^{\mathbf{b}} \bar{\epsilon}_{\beta}^{\mathbf{c}} \right) + \sqrt{2} C_{\mu\alpha\delta}^{\mathbf{b}} \epsilon_{\beta}^{\mathbf{c}\delta} - (\alpha \leftrightarrow \beta) \right] \\
& + \sqrt{2} \left[ T_{\mu\alpha}^{\mathbf{a}} \epsilon_{\beta} + \bar{T}_{\mu\alpha}^{\mathbf{a}} \bar{\epsilon}_{\beta} + T_{\mu\alpha} \epsilon_{\beta}^{\mathbf{a}} + \bar{T}_{\mu\alpha} \bar{\epsilon}_{\beta}^{\mathbf{a}} + 2 \left( C_{\mu\alpha\delta}^{\mathbf{a}} \epsilon_{\beta}^{\delta} + C_{\mu\alpha\delta} \epsilon_{\beta}^{\mathbf{a}\delta} \right) \right. \\
& \left. - (\alpha \leftrightarrow \beta) \right] \Big\}, \tag{62}
\end{aligned}$$

where the expressions for  $\delta\bar{A}_{\mu}$ ,  $\delta T_{\mu\nu}$ ,  $\delta\bar{T}_{\mu\nu}$ , and  $\delta C_{\mu\alpha\beta}$  are given in (16), and  $\bar{\epsilon}_{\alpha\beta} \equiv \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\gamma\delta}$ ,  $\bar{\epsilon}_{\alpha\beta}^{\mathbf{a}} \equiv \epsilon_{\alpha\beta\gamma\delta} \epsilon^{\mathbf{a}\gamma\delta}$ . (To avoid confusion we recall that  $\epsilon_{\alpha\beta\gamma\delta}$  is the complete antisymmetric tensor in Euclidean space.) In the lagrangian (60) (and therefore in the Feynman rules) the contractions of the type  $d^{\mathbf{abc}} f^{\mathbf{cef}}$ , which cannot be reduced as the product of combinations of  $\delta^{\mathbf{ab}}$ ,  $d^{\mathbf{abc}}$ , and  $f^{\mathbf{abc}}$  for a general  $SU(N)$  group [32], appear. (Note that simplifications can be obtained by choosing some particular internal groups). Clearly the physical amplitudes and correlation functions can be expressed as products of  $\delta^{\mathbf{ab}}$ . Useful results and techniques are provided in Refs.[33] for the calculations of the traces of the  $\mathbf{T}^{\mathbf{a}}$  matrix products.